1 №39

Условия задачи:

$$U_{tt} = U_{xx} + x\left(x-l\right)t^2 \tag{1}$$

$$U|_{t=0} = U_t|_{t=0} = 0 \tag{2}$$

$$U|_{x=0} = U|_{x=l} = 0 \tag{3}$$

Ищем решение в виде

$$U\left(x,t\right) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{l}, \tag{4}$$

$$b_n = \frac{2}{l}\int\limits_0^l U\left(z,t\right)\sin\frac{\pi n z}{l}dz, \tag{5}$$

$$b_n = b_n\left(t\right). \tag{6}$$

Разложим также

$$x\left(x-l\right)t^2 = \sum_{n=1}^{\infty}\beta_n \sin\frac{\pi n x}{l}\ : \tag{7}$$

$$\begin{gathered}\beta_n = \frac{2}{l}\int\limits_0^l z\left(z-l\right)t^2\sin\frac{\pi n z}{l}dz = -\frac{2t^2}{l}\frac{l}{\pi n}\int\limits_0^l z\left(z-l\right)\left(\cos\frac{\pi n z}{l}\right)' dz = \\ = -\frac{2t^2}{\pi n}z\left(z-l\right)\cos\frac{\pi n z}{l}\Bigg|_{z=0}^{z=l} + \frac{2t^2}{\pi n}\int\limits_0^l\left(z^2-lz\right)'\left(\cos\frac{\pi n z}{l}\right)dz = \frac{2t^2}{\pi n}\int\limits_0^l\left(2z-l\right)\cos\frac{\pi n z}{l}dz = \\ = \frac{2t^2}{\pi n}\frac{l}{\pi n}\int\limits_0^l\left(2z-l\right)\left(\sin\frac{\pi n z}{l}\right)' dz = \frac{2t^2 l}{\left(\pi n\right)^2}\left(2z-l\right)\sin\frac{\pi n z}{l}\Bigg|_{z=0}^{z=l} - \frac{2t^2 l}{\left(\pi n\right)^2}\int\limits_0^l\left(2z-l\right)'\left(\sin\frac{\pi n z}{l}\right)dz = \\ = -\frac{4t^2 l}{\left(\pi n\right)^2}\int\limits_0^l\sin\frac{\pi n z}{l}dz = \frac{4t^2 l}{\left(\pi n\right)^2}\frac{l}{\pi n}\cos\frac{\pi n z}{l}\Bigg|_{z=0}^{z=l} = \\ = \frac{4t^2 l^2}{\left(\pi n\right)^3}\left(\left(-1\right)^n - 1\right).\end{gathered} \tag{8}$$

Подставим в уравнение (1)

$$\frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{l} = \frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{l} + \sum_{n=1}^{\infty}\beta_n \sin\frac{\pi n x}{l} \tag{9}$$

$$\sum_{n=1}^{\infty} b_n'' \sin\frac{\pi n x}{l} = -\sum_{n=1}^{\infty}\left(\frac{\pi n}{l}\right)^2 b_n \sin\frac{\pi n x}{l} + \sum_{n=1}^{\infty}\beta_n \sin\frac{\pi n x}{l} \tag{10}$$

$$\sum_{n=1}^{\infty}\left[b_n'' + \left(\frac{\pi n}{l}\right)^2 b_n - \beta_n\right]\sin\frac{\pi n x}{l} = 0 \tag{11}$$

$$b_n'' + \left(\frac{\pi n}{l}\right)^2 b_n - \frac{4t^2 l^2}{(\pi n)^3}\left((-1)^n - 1\right) = 0 \tag{12}$$

$$b_n = b_n^0 + b_n^1 \tag{13}$$

Однородное:

$$b_n^{''0} + \left(\frac{\pi n}{l}\right)^2 b_n^0 = 0 \tag{14}$$

$$\lambda^2 + \left(\frac{\pi n}{l}\right)^2 = 0 \qquad \Longrightarrow \qquad \lambda = \pm i\frac{\pi n}{l} \tag{15}$$

$$b_n^0 = C_n^1 \cos\frac{\pi n t}{l} + C_n^2 \sin\frac{\pi n t}{l} \tag{16}$$

Частное решение неоднородного:

$$b_n^1 = c_2 t^2 + c_1 t + c_0. \tag{17}$$

Подставим его в (12):

$$2c_2 + \left(\frac{\pi n}{l}\right)^2 \left(c_2 t^2 + c_1 t + c_0\right) - \frac{4t^2 l^2}{(\pi n)^3}\left((-1)^n - 1\right) = 0 \tag{18}$$

$$\begin{cases} \left(\frac{\pi n}{l}\right)^2 c_2 - \frac{4l^2}{(\pi n)^3}\left((-1)^n - 1\right) = 0 \\ c_1 = 0 \\ 2c_2 + \left(\frac{\pi n}{l}\right)^2 c_0 = 0 \end{cases} \tag{19}$$

$$c_2 = \frac{4l^4}{(\pi n)^5}\left((-1)^n - 1\right) \tag{20}$$

$$c_0 = -2c_2\frac{l^2}{\pi^2 n^2} = -\frac{8l^6}{(\pi n)^7}\left((-1)^n - 1\right) \tag{21}$$

Подставим в b_n^1:

$$b_n^1 = \frac{4l^4 t^2}{(\pi n)^5}\left((-1)^n - 1\right) - \frac{8l^6}{(\pi n)^7}\left((-1)^n - 1\right) = \frac{4l^4\left((-1)^n - 1\right)}{(\pi n)^7}\left[(\pi n)^2 t^2 - 2l^2\right] \tag{22}$$

Найдём b_n:

$$b_n = b_n^0 + b_n^1 = C_n^1 \cos\frac{\pi n t}{l} + C_n^2 \sin\frac{\pi n t}{l} + \frac{4l^4\left((-1)^n - 1\right)}{(\pi n)^7}\left[(\pi n)^2 t^2 - 2l^2\right] \tag{23}$$

Из (4)

$$U(x,t) = \sum_{n=1}^{\infty}\left\{C_n^1 \cos\frac{\pi n t}{l} + C_n^2 \sin\frac{\pi n t}{l} + \frac{4l^4\left((-1)^n - 1\right)}{(\pi n)^7}\left[(\pi n)^2 t^2 - 2l^2\right]\right\}\sin\frac{\pi n x}{l} \tag{24}$$

Теперь учтём начальные условия:

$$U_t\big|_{t=0} = \sum_{n=1}^{\infty}\left\{\frac{\pi n}{l}C_n^2\right\}\sin\frac{\pi n x}{l} = 0 \tag{25}$$

$$\frac{\pi n}{l}C_n^2 = 0 \quad \Longrightarrow \quad C_n^2 = 0 \tag{26}$$

$$U|_{t=0} = \sum_{n=1}^{\infty} \left\{ C_n^1 - \frac{8l^6\left((-1)^n - 1\right)}{(\pi n)^7} \right\} \sin\frac{\pi n x}{l} = 0 \tag{27}$$

$$C_n^1 - \frac{8l^6\left((-1)^n - 1\right)}{(\pi n)^7} = 0 \quad \Longrightarrow \quad C_n^1 = \frac{8l^6\left((-1)^n - 1\right)}{(\pi n)^7} \tag{28}$$

Окончательный ответ:

$$\begin{gathered} U(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{8l^6\left((-1)^n - 1\right)}{(\pi n)^7} \cos\frac{\pi n t}{l} + \frac{4l^4\left((-1)^n - 1\right)}{(\pi n)^7} \left[(\pi n)^2 t^2 - 2l^2 \right] \right\} \sin\frac{\pi n x}{l} = \\ = \sum_{n=1}^{\infty} \frac{4l^4}{(\pi n)^7} \left((-1)^n - 1\right) \left\{ 2l^2 \cos\frac{\pi n t}{l} + (\pi n)^2 t^2 - 2l^2 \right\} \sin\frac{\pi n x}{l} = \\ = \sum_{k=1}^{\infty} \frac{-8l^4}{\pi^7 (2k-1)^7} \left\{ 2l^2 \cos\frac{\pi(2k-1)t}{l} + \pi^2(2k-1)^2 t^2 - 2l^2 \right\} \sin\frac{\pi(2k-1)x}{l}. \end{gathered} \tag{29}$$

2 Неоднородные граничные условия

$$U|_{x=0} = \varphi(t) \quad U|_{x=l} = \psi(t) \tag{30}$$

$$U = V + \frac{l-x}{l}\varphi(t) + \frac{x}{l}\psi(t) \tag{31}$$

$$V|_{x=0} = 0 \quad V|_{x=l} = 0 \tag{32}$$