1 Условия задачи №52

$$U\left(x,y,t\right), \quad 0\leqslant x\leqslant l, \quad 0\leqslant y\leqslant m, \quad m=l$$

$$U_{tt}=a^2\left(U_{xx}+U_{yy}\right) \tag{1}$$

$$\left.U\right|_{t=0}=\frac{l}{100}\sin\frac{\pi x}{l}\sin\frac{\pi y}{l} \tag{2}$$

$$\left.U_t\right|_{t=0}=0 \tag{3}$$

$$\left.U\right|_{x=0}=\left.U\right|_{x=l}=0 \tag{4}$$

$$\left.U\right|_{y=0}=\left.U\right|_{y=l}=0 \tag{5}$$

2 Разложение функции U в двумерный ряд Фурье

$$U\left(x,y,t\right)=\sum_{k=1}^{\infty}a_k\sin\frac{\pi kx}{l} \tag{6}$$

$$a_k=\frac{2}{l}\int\limits_0^l U\left(z,y,t\right)\sin\frac{\pi kz}{l}dz=a_k\left(y,t\right) \tag{7}$$

$$a_k=\sum_{n=1}^{\infty}b_{nk}\sin\frac{\pi ny}{m}, \tag{8}$$

$$b_{nk}=\frac{2}{m}\int\limits_0^m a_k\left(s,t\right)\sin\frac{\pi ns}{m}ds=b_{nk}\left(t\right) \tag{9}$$

$$U\left(x,y,t\right)=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}b_{nk}\sin\frac{\pi ny}{m}\sin\frac{\pi kx}{l} \tag{10}$$

$$b_{nk}=\frac{4}{lm}\int\limits_0^m ds\int\limits_0^l dzU\left(z,s,t\right)\sin\frac{\pi kz}{l}\sin\frac{\pi ns}{m} \tag{11}$$

При $m=l$ условия (4) и (5) сразу выполняются.

3 Решение

Подставим (10) в уравнение (1) и найдём функции $b_{nk}(t)$:

$$\frac{\partial^2}{\partial t^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b_{nk}\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} = a^2\left(\frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b_{nk}\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} + \frac{\partial^2}{\partial y^2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b_{nk}\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l}\right), \tag{12}$$

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b''_{nk}\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} = a^2\left(-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b_{nk}\sin\frac{\pi ny}{l}\left(\frac{\pi k}{l}\right)^2\sin\frac{\pi kx}{l} - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} b_{nk}\left(\frac{\pi n}{l}\right)^2\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l}\right), \tag{13}$$

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[b''_{nk} + a^2 b_{nk}\left(\frac{\pi}{l}\right)^2\left(k^2+n^2\right)\right]\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} = 0, \tag{14}$$

$$b''_{nk} + \left(\frac{\pi a}{l}\right)^2\left(k^2+n^2\right)b_{nk} = 0, \tag{15}$$

$$b_{nk} = C^1_{nk}\cos\left(\sqrt{k^2+n^2}\frac{\pi at}{l}\right) + C^2_{nk}\sin\left(\sqrt{k^2+n^2}\frac{\pi at}{l}\right). \tag{16}$$

$$U = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[C^1_{nk}\cos\left(\sqrt{k^2+n^2}\frac{\pi at}{l}\right) + C^2_{nk}\sin\left(\sqrt{k^2+n^2}\frac{\pi at}{l}\right)\right]\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} \tag{17}$$

Определяем константы C^1_{nk} и C^2_{nk} из начальных условий. Из (3)

$$U_t|_{t=0} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\sqrt{k^2+n^2}\frac{\pi a}{l}C^2_{nk}\right]\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} = 0 \tag{18}$$

найдём

$$\sqrt{k^2+n^2}\frac{\pi a}{l}C^2_{nk} = 0 \quad\Longrightarrow\quad C^2_{nk} = 0; \tag{19}$$

а из (2)

$$U|_{t=0} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} C^1_{nk}\sin\frac{\pi ny}{l}\sin\frac{\pi kx}{l} = \frac{l}{100}\sin\frac{\pi x}{l}\sin\frac{\pi y}{l} \tag{20}$$

найдём

$$C^1_{11} = \frac{l}{100};\quad C^1_{nk} = 0,\quad n,k \neq 1. \tag{21}$$

Подставляем константы:

$$U = \frac{l}{100}\cos\frac{\sqrt{2}\pi at}{l}\sin\frac{\pi y}{l}\sin\frac{\pi x}{l}. \tag{22}$$