Неопределённые интегралы с одним параметром

$$\int e^x \sin(\gamma x)\,dx = e^x \sin(\gamma x) - \gamma \int e^x \cos(\gamma x)\,dx = e^x \sin(\gamma x) - \gamma e^x \cos(\gamma x) - \gamma^2 \int e^x \sin(\gamma x)\,dx \tag{1}$$

$$\int e^x \sin(\gamma x)\,dx = \frac{1}{1+\gamma^2} e^x \sin(\gamma x) - \frac{\gamma}{1+\gamma^2} e^x \cos(\gamma x) + C \tag{2}$$

$$\int e^x \cos(\gamma x)\,dx = -\frac{1}{\gamma}\int e^x \sin(\gamma x)\,dx + \frac{1}{\gamma} e^x \sin(\gamma x) = -\frac{1}{\gamma}\frac{1}{1+\gamma^2} e^x \sin(\gamma x) + \frac{1}{1+\gamma^2} e^x \cos(\gamma x) - \frac{C}{\gamma} + \frac{1}{\gamma} e^x \sin(\gamma x) = \tag{3}$$

$$= \frac{\gamma}{1+\gamma^2} e^x \sin(\gamma x) + \frac{1}{1+\gamma^2} e^x \cos(\gamma x) - \frac{C}{\gamma}$$

Неопределённые интегралы с двумя параметрами

$$\int e^{-\alpha x} \sin(\beta x)\,dx =$$

$-\alpha x = y$, $x = -\frac{y}{\alpha}$

$$= \int e^y \sin\left(\frac{y}{\alpha}\beta\right) d\frac{y}{\alpha} = \frac{1}{\alpha}\int e^y \sin\left(\frac{\beta}{\alpha} y\right) dy = \tag{4}$$

$\gamma = \frac{\beta}{\alpha}$

$$= \frac{1}{\alpha}\left[\frac{1}{1+\left(\frac{\beta}{\alpha}\right)^2} e^y \sin\left(\frac{\beta}{\alpha} y\right) - \frac{\frac{\beta}{\alpha}}{1+\left(\frac{\beta}{\alpha}\right)^2} e^y \cos\left(\frac{\beta}{\alpha} y\right) + C\right] = -\frac{\alpha}{\alpha^2+\beta^2} e^{-\alpha x}\sin(\beta x) - \frac{\beta}{\alpha^2+\beta^2} e^{-\alpha x}\cos(\beta x) + \frac{C}{\alpha}$$

Аналогично,

$$\int e^{-\alpha x}\cos(\beta x)\,dx = -\frac{1}{\alpha}\int e^y \cos\left(\frac{\beta}{\alpha} y\right) dy = -\frac{1}{\alpha}\left[\frac{\frac{\beta}{\alpha}}{1+\left(\frac{\beta}{\alpha}\right)^2} e^y \sin\left(\frac{\beta}{\alpha} y\right) + \frac{1}{1+\left(\frac{\beta}{\alpha}\right)^2} e^y \cos\left(\frac{\beta}{\alpha} y\right) - \frac{\alpha C}{\beta}\right] = \tag{5}$$

$$= -\frac{\beta}{\alpha^2+\beta^2} e^y \sin\left(\frac{\beta}{\alpha} y\right) - \frac{\alpha}{\alpha^2+\beta^2} e^y \cos\left(\frac{\beta}{\alpha} y\right) + \frac{C}{\beta} = \frac{\beta}{\alpha^2+\beta^2} e^{-\alpha x}\sin(\beta x) - \frac{\alpha}{\alpha^2+\beta^2} e^{-\alpha x}\cos(\beta x) + \frac{C}{\beta}$$

Несобственные интегралы

$$\int\limits_0^\infty e^{-\alpha x}\sin(\beta x)\,dx = -\frac{\alpha}{\alpha^2+\beta^2} e^{-\alpha x}\sin(\beta x) - \frac{\beta}{\alpha^2+\beta^2} e^{-\alpha x}\cos(\beta x)\Big|_0^\infty = \frac{\beta}{\alpha^2+\beta^2} \tag{6}$$

$$\int\limits_0^\infty e^{-\alpha x}\cos(\beta x)\,dx = \frac{\beta}{\alpha^2+\beta^2} e^{-\alpha x}\sin(\beta x) - \frac{\alpha}{\alpha^2+\beta^2} e^{-\alpha x}\cos(\beta x)\Big|_0^\infty = \frac{\alpha}{\alpha^2+\beta^2} \tag{7}$$

Пусть

$$I(\alpha,\beta) = \int\limits_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}dx \tag{8}$$

$$I'_\alpha = -\int\limits_0^\infty e^{-\alpha x}\sin(\beta x)\,dx = -\frac{\beta}{\alpha^2+\beta^2} \tag{9}$$

$$I'_\beta = \int\limits_0^\infty e^{-\alpha x}\cos(\beta x)\,dx = \frac{\alpha}{\alpha^2+\beta^2} \tag{10}$$

$$I = -\int \frac{\beta}{\alpha^2+\beta^2}d\alpha = -\int \frac{\frac{1}{\beta}}{\frac{\alpha^2}{\beta^2}+1}d\alpha = -\operatorname{arctg}\frac{\alpha}{\beta} + \varphi(\beta) \tag{11}$$

$$I'_\beta = \frac{1}{1+\left(\frac{\alpha}{\beta}\right)^2}\frac{\alpha}{\beta^2} + \varphi'(\beta) = \frac{\alpha}{\beta^2+\alpha^2} + \varphi'(\beta) = \frac{\alpha}{\alpha^2+\beta^2} \tag{12}$$

$$\varphi'(\beta) = 0, \qquad \varphi(\beta) = C \tag{13}$$

$$I = \int\limits_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}dx = -\operatorname{arctg}\frac{\alpha}{\beta} + C \tag{14}$$

$$\lim_{\beta\to+0} I = \int\limits_0^\infty e^{-\alpha x}\frac{\sin(0)}{x}dx = 0 = -\frac{\pi}{2} + C, \quad C = \frac{\pi}{2} \tag{15}$$

$$\lim_{\beta\to-0} I = 0 = \frac{\pi}{2} + C, \quad C = -\frac{\pi}{2} \tag{16}$$

$$C = \frac{\pi}{2}\operatorname{sgn}\beta \tag{17}$$

$$I = \int\limits_0^\infty e^{-\alpha x}\frac{\sin(\beta x)}{x}dx = -\operatorname{arctg}\frac{\alpha}{\beta} + \frac{\pi}{2}\operatorname{sgn}\beta \tag{18}$$

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$$\lim_{\alpha\to+0} I = \int\limits_0^\infty \frac{\sin(\beta x)}{x}dx = \frac{\pi}{2}\operatorname{sgn}\beta \tag{19}$$