$$\int x^p\left(ax^q+b\right)^r dx, \qquad p,q,r\in\mathbb{Q} \tag{1}$$

1) $r\in\mathbb{Z}$: тогда

$$x=z^n \quad (pn\in\mathbb{Z},\ qn\in\mathbb{Z}), dx=nz^{n-1}dz \tag{2}$$

$$\int x^p\left(ax^q+b\right)^r dx=\int z^{np}\left(az^{nq}+b\right)^r nz^{n-1}dz=n\int z^{np+n-1}\left(az^{nq}+b\right)^r dz \tag{3}$$

2)

$$ax^q+b=z^n, \qquad x=\left(\frac{z^n-b}{a}\right)^{\frac{1}{q}}, \qquad dx=\frac{1}{q}\left(\frac{z^n-b}{a}\right)^{\frac{1}{q}-1}\frac{nz^{n-1}}{a}dz=\frac{n}{aq}\left(\frac{z^n-b}{a}\right)^{\frac{1}{q}-1}z^{n-1}dz \tag{4}$$

$n\in\mathbb{N}$

$$\int x^p\left(ax^q+b\right)^r dx=\int\left(\frac{z^n-b}{a}\right)^{\frac{1}{q}p}z^{nr}\frac{n}{aq}\left(\frac{z^n-b}{a}\right)^{\frac{1}{q}-1}z^{n-1}dz=$$

$$=\frac{n}{aq}\int\left(\frac{z^n-b}{a}\right)^{\frac{p+1}{q}-1}z^{nr+n-1}dz \tag{5}$$

Применяем, если $\frac{p+1}{q}\in\mathbb{Z}$. Подбираем $nr\in\mathbb{Z}$.

3)

$$\int x^p\left(ax^q+b\right)^r dx=\int x^{p+qr}\left(a+bx^{-q}\right)^r dx \tag{6}$$

$$a+bx^{-q}=z^n, \qquad x=\left(\frac{z^n-a}{b}\right)^{-\frac{1}{q}}, \qquad dx=-\frac{n}{qb}\left(\frac{z^n-a}{b}\right)^{-\frac{1}{q}-1}z^{n-1}dz \tag{7}$$

$$\int x^p\left(ax^q+b\right)^r dx=\int x^{p+qr}\left(a+bx^{-q}\right)^r dx=-\frac{n}{qb}\int\left(\frac{z^n-a}{b}\right)^{-\frac{p+1}{q}-r-1}z^{nr+n-1}dz \tag{8}$$

Применяем, если $\frac{p+1}{q}+r\in\mathbb{Z}$. Подбираем $nr\in\mathbb{Z}$.

Демидович, №1982

$$\int\frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}dx=\int x^{\frac{1}{2}}\left(1+x^{\frac{1}{3}}\right)^{-2}dx \tag{9}$$

$r=2\in\mathbb{Z}$ – случай 1

$$x=z^6, \quad dx=6z^5dz, \tag{10}$$

$$z^8=\left(z^2\right)^4=\left(\left(1+z^2\right)-1\right)^4=\left(1+z^2\right)^4-4\left(1+z^2\right)^3+6\left(1+z^2\right)^2-4\left(1+z^2\right)+1; \tag{11}$$

$$\int\frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}dx=\int x^{\frac{1}{2}}\left(1+x^{\frac{1}{3}}\right)^{-2}dx=6\int z^3\left(1+z^2\right)^{-2}z^5dz=6\int\frac{z^8}{\left(1+z^2\right)^2}dz=$$

$$=6\int\left[\left(1+z^2\right)^2-4\left(1+z^2\right)+6-4\frac{1}{1+z^2}+\frac{1}{\left(1+z^2\right)^2}\right]dz=6\int\left[3-2z^2+z^4-4\frac{1}{1+z^2}+\frac{2z}{2z\left(1+z^2\right)^2}\right]dz= \tag{12}$$

$$=6\left(3z-\frac{2}{3}z^3+\frac{z^5}{5}-4\operatorname{arctg}z\right)-3\int\frac{1}{z}\left(\frac{1}{1+z^2}\right)'dz.$$

$$\int\frac{1}{z}\left(\frac{1}{1+z^2}\right)'dz=\frac{1}{z}\frac{1}{1+z^2}-\int\left(\frac{1}{z}\right)'\frac{1}{1+z^2}dz=\frac{1}{z\left(1+z^2\right)}+\int\frac{1}{z^2\left(1+z^2\right)}dz=$$

$$=\frac{1}{z\left(1+z^2\right)}+\int\left(\frac{1}{z^2}-\frac{1}{1+z^2}\right)dz=\frac{1}{z\left(1+z^2\right)}-\frac{1}{z}-\operatorname{arctg}z+C_1, \tag{13}$$

$$\int\frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}dx=6\left(3z-\frac{2}{3}z^3+\frac{z^5}{5}-4\operatorname{arctg}z\right)-3\int\frac{1}{z}\left(\frac{1}{1+z^2}\right)'dz=$$

$$=18z-4z^3+\frac{6}{5}z^5-24\operatorname{arctg}z-3\frac{1}{z\left(1+z^2\right)}+3\frac{1}{z}+3\operatorname{arctg}z-3C_1= \tag{14}$$

$$=18z-4z^3+\frac{6}{5}z^5+\frac{3}{z}-\frac{3}{z\left(1+z^2\right)}-21\operatorname{arctg}z+C.$$

Обратно: $z = x^{\frac{1}{6}}$

$$\int \frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}dx = 18z - 4z^3 + \frac{6}{5}z^5 + \frac{3}{z} - \frac{3}{z\left(1+z^2\right)} - 21\operatorname{arctg} z + C =$$
$$= 18x^{\frac{1}{6}} - 4x^{\frac{1}{2}} + \frac{6}{5}x^{\frac{5}{6}} + 3x^{-\frac{1}{6}} - \frac{3}{x^{\frac{1}{6}}\left(1+x^{\frac{1}{3}}\right)} - 21\operatorname{arctg}\left(x^{\frac{1}{6}}\right) + C. \tag{15}$$

Демидович, №1981

$$\int \sqrt{x^3+x^4}dx = \int x^{\frac{3}{2}}\left(1+x\right)^{\frac{1}{2}}dx \tag{16}$$

$p = \frac{3}{2}$, $q = 1$, $r = \frac{1}{2}$

$$\frac{p+1}{q} = \frac{\frac{3}{2}+1}{1} = \frac{5}{2}$$
$$\frac{p+1}{q} + r = \frac{5}{2} + \frac{1}{2} = 3 \in \mathbb{Z}$$

случай 3

$$z = \sqrt{x^{-1}+1}, \quad x^{-1}+1 = z^2, \quad x = \frac{1}{z^2-1}, \quad dx = -\frac{2zdz}{\left(z^2-1\right)^2} \tag{17}$$

$$\int \sqrt{x^3+x^4}dx = \int x^{\frac{3}{2}}\left(1+x\right)^{\frac{1}{2}}dx = \int x^2\left(x^{-1}+1\right)^{\frac{1}{2}}dx = -2\int\left(\frac{1}{z^2-1}\right)^2 z\frac{zdz}{\left(z^2-1\right)^2} = -2\int\frac{z^2}{\left(z^2-1\right)^4}dz \tag{18}$$

По частям 1:

$$\left(\frac{1}{\left(z^2-1\right)^3}\right)' = -6\frac{z}{\left(z^2-1\right)^4}, \quad \left(\frac{1}{\left(z^2-1\right)^2}\right)' = -4\frac{z}{\left(z^2-1\right)^3}. \tag{19}$$

$$\int\frac{z^2}{\left(z^2-1\right)^4}dz = -\frac{1}{6}\int z\frac{-6z}{\left(z^2-1\right)^4}dz = -\frac{1}{6}\int z\left(\frac{1}{\left(z^2-1\right)^3}\right)'dz = -\frac{1}{6}z\frac{1}{\left(z^2-1\right)^3} + \frac{1}{6}\int\frac{dz}{\left(z^2-1\right)^3} =$$
$$= -\frac{1}{6}z\frac{1}{\left(z^2-1\right)^3} + \frac{1}{6}\int\frac{1-z^2+z^2}{\left(z^2-1\right)^3}dz = -\frac{1}{6}z\frac{1}{\left(z^2-1\right)^3} - \frac{1}{6}\int\frac{1}{\left(z^2-1\right)^2}dz + \frac{1}{6}\int\frac{z^2}{\left(z^2-1\right)^3}dz; \tag{20}$$

По частям 2:

$$\int\frac{z^2}{\left(z^2-1\right)^3}dz = -\frac{1}{4}\int z\frac{-4z}{\left(z^2-1\right)^3}dz = -\frac{1}{4}\int z\left(\frac{1}{\left(z^2-1\right)^2}\right)'dz = -\frac{1}{4}\frac{z}{\left(z^2-1\right)^2} + \frac{1}{4}\int\frac{1}{\left(z^2-1\right)^2}dz,$$

$$\int\frac{z^2}{\left(z^2-1\right)^4}dz = -\frac{1}{6}z\frac{1}{\left(z^2-1\right)^3} - \frac{1}{6}\int\frac{1}{\left(z^2-1\right)^2}dz + \frac{1}{6}\int\frac{z^2}{\left(z^2-1\right)^3}dz =$$
$$= -\frac{1}{6}z\frac{1}{\left(z^2-1\right)^3} - \frac{1}{6}\int\frac{1}{\left(z^2-1\right)^2}dz + \frac{1}{6}\left[-\frac{1}{4}\frac{z}{\left(z^2-1\right)^2} + \frac{1}{4}\int\frac{1}{\left(z^2-1\right)^2}dz\right] = \tag{21}$$
$$= -\frac{1}{24}z\frac{4}{\left(z^2-1\right)^3} - \frac{1}{24}z\frac{1}{\left(z^2-1\right)^2} - \frac{1}{8}\int\frac{1}{\left(z^2-1\right)^2}dz = -\frac{1}{24}z\frac{z^2+3}{\left(z^2-1\right)^3} + \frac{1}{8}\int\frac{1}{z^2-1}dz - \frac{1}{8}\int\frac{z^2}{\left(z^2-1\right)^2}dz.$$

По частям 3:

$$\left(\frac{1}{z^2-1}\right)' = \frac{-2z}{\left(z^2-1\right)^2}, \tag{22}$$

$$\int\frac{z^2}{\left(z^2-1\right)^2}dz = -\frac{1}{2}\int z\frac{-2z}{\left(z^2-1\right)^2}dz = -\frac{1}{2}\int z\left(\frac{1}{z^2-1}\right)'dz = -\frac{1}{2}\frac{z}{z^2-1} + \frac{1}{2}\int\frac{1}{z^2-1}dz, \tag{23}$$

$$\int\frac{z^2}{\left(z^2-1\right)^4}dz = -\frac{1}{24}z\frac{z^2+3}{\left(z^2-1\right)^3} + \frac{1}{8}\int\frac{1}{z^2-1}dz - \frac{1}{8}\int\frac{z^2}{\left(z^2-1\right)^2}dz =$$
$$= -\frac{1}{24}z\frac{z^2+3}{\left(z^2-1\right)^3} + \frac{1}{8}\int\frac{1}{z^2-1}dz - \frac{1}{8}\left[-\frac{1}{2}\frac{z}{z^2-1} + \frac{1}{2}\int\frac{1}{z^2-1}dz\right] = \tag{24}$$
$$= -\frac{z}{24}\frac{z^2+3}{\left(z^2-1\right)^3} + \frac{1}{16}\frac{z}{z^2-1} + \frac{1}{16}\int\frac{1}{z^2-1}dz = -\frac{z}{24}\frac{z^2+3}{\left(z^2-1\right)^3} + \frac{1}{16}\frac{z}{z^2-1} + \frac{1}{32}\ln\left|\frac{1-z}{1+z}\right| + C.$$

Окончательно,

$$\int \sqrt{x^3+x^4}dx = -2\int \frac{z^2}{\left(z^2-1\right)^4}dz = \frac{z}{12}\frac{z^2+3}{\left(z^2-1\right)^3} - \frac{1}{8}\frac{z}{z^2-1} - \frac{1}{16}\ln\left|\frac{1-z}{1+z}\right| + C =$$

$$= \frac{z}{24}\left[2\frac{z^2+3}{\left(z^2-1\right)^3} - \frac{3}{z^2-1}\right] - \frac{1}{16}\ln\left|\frac{1-z}{1+z}\right| + C = \tag{25}$$

$z=\sqrt{x^{-1}+1}$, $\frac{1}{z^2-1}=x$

$$= \frac{\sqrt{x^{-1}+1}}{24}\left[2\left(x^{-1}+4\right)x^3-3x\right] - \frac{1}{16}\ln\left|\frac{1-\sqrt{x^{-1}+1}}{1+\sqrt{x^{-1}+1}}\right| + C = \frac{\sqrt{x^{-1}+1}}{24}\left(8x^3+2x^2-3x\right) - \frac{1}{16}\ln\left|\frac{\left(1-\sqrt{x^{-1}+1}\right)^2}{-x^{-1}}\right| + C =$$

$$= \frac{\sqrt{x^{-1}+1}}{24}\left(8x^3+2x^2-3x\right) - \frac{1}{8}\ln\left|1-\sqrt{x^{-1}+1}\right| - \frac{1}{16}\ln|x| + C.$$