$$\frac{\partial}{\partial y}e^{-y\left(1+x^2\right)} = -\left(1+x^2\right)e^{-y\left(1+x^2\right)}, \tag{1}$$

$$\frac{\partial}{\partial y}\frac{e^{-y\left(1+x^2\right)}}{1+x^2} = -e^{-y\left(1+x^2\right)}, \tag{2}$$

$$\int\limits_0^\infty e^{-y\left(1+x^2\right)}dy = -\left.\frac{e^{-y\left(1+x^2\right)}}{1+x^2}\right|_0^\infty = -\left(0-\frac{1}{1+x^2}\right) = \frac{1}{1+x^2}. \tag{3}$$

Теперь перейдём к основному интегралу.

$$L\left(\alpha\right) = \int\limits_0^\infty \frac{\cos\alpha x}{1+x^2}dx = \int\limits_0^\infty \cos\alpha x\int\limits_0^\infty e^{-y\left(1+x^2\right)}dydx = \int\limits_0^\infty\int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dxdy \tag{4}$$

Введём интеграл с двумя параметрами

$$I\left(\alpha, y\right) = \int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dx, \tag{5}$$

$$I'_\alpha = -\int\limits_0^\infty e^{-y\left(1+x^2\right)}x\sin\alpha x\,dx, \tag{6}$$

$$\frac{\partial}{\partial x}e^{-y\left(1+x^2\right)} = -ye^{-y\left(1+x^2\right)}\frac{\partial}{\partial x}\left(1+x^2\right) = -2xye^{-y\left(1+x^2\right)}, \tag{7}$$

$$I'_\alpha = \frac{1}{2y}\int\limits_0^\infty\left(-2xye^{-y\left(1+x^2\right)}\right)\sin\alpha x\,dx = \frac{1}{2y}\int\limits_0^\infty\left(e^{-y\left(1+x^2\right)}\right)'\sin\alpha x\,dx = \frac{1}{2y}e^{-y\left(1+x^2\right)}\sin\alpha x\Big|_0^\infty -$$

$$-\frac{1}{2y}\int\limits_0^\infty\left(\sin\alpha x\right)'e^{-y\left(1+x^2\right)}dx = -\frac{\alpha}{2y}\int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dx = -\frac{\alpha}{2y}I. \tag{8}$$

Дифференциальное уравнение:

$$\frac{I'_\alpha}{I} = -\frac{\alpha}{2y}. \tag{9}$$

$$\ln|I| = -\frac{\alpha^2}{4y} + \tilde{\varphi}\left(y\right), \tag{10}$$

$$|I| = e^{-\frac{\alpha^2}{4y}+\tilde{\varphi}(y)} = e^{\tilde{\varphi}(y)}e^{-\frac{\alpha^2}{4y}}, \tag{11}$$

$$I = \pm e^{\tilde{\varphi}(y)}e^{-\frac{\alpha^2}{4y}} = \varphi\left(y\right)e^{-\frac{\alpha^2}{4y}}. \tag{12}$$

$\alpha = 0$:

$$I\left(0, y\right) = \varphi\left(y\right) = \int\limits_0^\infty e^{-y\left(1+x^2\right)}dx = e^{-y}\int\limits_0^\infty e^{-yx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-y}, \tag{13}$$

$$I=\varphi\left(y\right)e^{-\frac{\alpha^2}{4y}}=\frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-y}e^{-\frac{\alpha^2}{4y}}=\frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-\left(y+\frac{\alpha^2}{4y}\right)}. \tag{14}$$

Вернёмся к основной задаче.

$$L\left(\alpha\right)=\int\limits_0^\infty\int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dxdy=\int\limits_0^\infty Idy=\int\limits_0^\infty\frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-\left(y+\frac{\alpha^2}{4y}\right)}dy. \tag{15}$$

$\sqrt{y}=z$

$$L\left(\alpha\right)=\sqrt{\pi}\int\limits_0^\infty e^{-\left(y+\frac{\alpha^2}{4y}\right)}\frac{dy}{2\sqrt{y}}=\sqrt{\pi}\int\limits_0^\infty e^{-\left(z^2+\frac{\alpha^2}{4z^2}\right)}dz \tag{16}$$

$$\left(z-\frac{\alpha}{2z}\right)^2=z^2+\frac{\alpha^2}{4z^2}-\alpha \tag{17}$$

$$L\left(\alpha\right)=\sqrt{\pi}\int\limits_0^\infty e^{-\left(z^2+\frac{\alpha^2}{4z^2}\right)}dz=\sqrt{\pi}\int\limits_0^\infty e^{-\left(z^2+\frac{\alpha^2}{4z^2}-\alpha+\alpha\right)}dz=\sqrt{\pi}e^{-\alpha}\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}dz \tag{18}$$

Введём ещё один вспомогательный интеграл:

$$J\left(\alpha\right)=\frac{L\left(\alpha\right)}{\sqrt{\pi}e^{-\alpha}}=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}dz \tag{19}$$

$$J'=\int\limits_0^\infty\frac{\partial}{\partial\alpha}e^{-\left(z-\frac{\alpha}{2z}\right)^2}dz=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}\left[-2\left(z-\frac{\alpha}{2z}\right)\left(-\frac{1}{2z}\right)\right]dz=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}\left(1-\frac{\alpha}{2z^2}\right)dz \tag{20}$$

$$\frac{\partial}{\partial z}\left(z-\frac{\alpha}{2z}\right)=\left(1+\frac{\alpha}{2z^2}\right) \tag{21}$$

$\alpha>0$:

$$\lim_{z\to\infty}\left(z-\frac{\alpha}{2z}\right)=\infty,\qquad\lim_{z\to+0}\left(z-\frac{\alpha}{2z}\right)=-\infty \tag{22}$$

$$2J-J'=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}2dz-\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}\left(1-\frac{\alpha}{2z^2}\right)dz=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}\left(1+\frac{\alpha}{2z^2}\right)dz=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}\frac{\partial}{\partial z}\left(z-\frac{\alpha}{2z}\right)dz \tag{23}$$

$z-\frac{\alpha}{2z}=u$

$$2J-J'=\int\limits_0^\infty e^{-\left(z-\frac{\alpha}{2z}\right)^2}\frac{\partial}{\partial z}\left(z-\frac{\alpha}{2z}\right)dz=\int\limits_{-\infty}^\infty e^{-u^2}du=\sqrt{\pi} \tag{24}$$

$$J'-2J=-\sqrt{\pi} \tag{25}$$

– линейное уравнение.

$$\left(e^{2\alpha}\right)'-2e^{2\alpha}=0\quad\Longrightarrow\quad J=e^{2\alpha}J_1, \tag{26}$$

$$e^{2\alpha}J_1'=-\sqrt{\pi} \tag{27}$$

$$J_1' = -e^{-2\alpha}\sqrt{\pi} \tag{28}$$

$$J_1 = \frac{\sqrt{\pi}}{2}e^{-2\alpha} + C \tag{29}$$

$$J = e^{2\alpha}\left(\frac{\sqrt{\pi}}{2}e^{-2\alpha} + C\right) = \frac{\sqrt{\pi}}{2} + e^{2\alpha}C \tag{30}$$

Найдём C:

$$J\left(0\right) = \frac{\sqrt{\pi}}{2} + C = \int\limits_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2}, \qquad C = 0 \tag{31}$$

$$J = \frac{\sqrt{\pi}}{2} \tag{32}$$

$$\frac{L\left(\alpha\right)}{\sqrt{\pi}e^{-\alpha}} = \frac{\sqrt{\pi}}{2} \tag{33}$$

$$L\left(\alpha\right) = \frac{\pi}{2}e^{-\alpha} = \frac{\pi}{2}e^{-|\alpha|} \tag{34}$$

$(\alpha > 0)$. $\alpha = 0$:

$$L\left(0\right) = \left.\int\limits_0^\infty \frac{\cos\alpha x}{1+x^2}dx\right|_{\alpha=0} = \int\limits_0^\infty \frac{1}{1+x^2}dx = \operatorname{arctg} x\big|_0^\infty = \frac{\pi}{2} = \left.\frac{\pi}{2}e^{-|\alpha|}\right|_{\alpha=0} \tag{35}$$

$\alpha < 0$:

$$L\left(\alpha\right) = \int\limits_0^\infty \frac{\cos\alpha x}{1+x^2}dx = \int\limits_0^\infty \frac{\cos\left(-\alpha x\right)}{1+x^2}dx = \frac{\pi}{2}e^{-(-\alpha)} = \frac{\pi}{2}e^{-|\alpha|} \tag{36}$$