$$\sum_{k=0}^{n} a_k y^{(k)} = f\left(x\right) \tag{1}$$

n частных решений однородного уравнения:

$$\sum_{k=0}^{n} a_k y_{0m}^{(k)} = 0, \tag{2}$$

$$y_0 = \sum_{m=1}^{n} C_m y_{0m}, \qquad C_m = const. \tag{3}$$

$$y = \sum_{m=1}^{n} \varphi_m\left(x\right) y_{0m}. \tag{4}$$

производная:

$$y' = \sum_{m=1}^{n} \varphi_m'\left(x\right) y_{0m} + \sum_{m=1}^{n} \varphi_m\left(x\right) y_{0m}'. \tag{5}$$

Первое условие на $\varphi_m\left(x\right)$:

$$\sum_{m=1}^{n} \varphi_m'\left(x\right) y_{0m} = 0, \qquad \Longrightarrow \qquad y' = \sum_{m=1}^{n} \varphi_m\left(x\right) y_{0m}'. \tag{6}$$

Аналогично:

$$\sum_{m=1}^{n} \varphi_m' y_{0m}' = 0 \qquad \Longrightarrow \qquad y'' = \sum_{m=1}^{n} \varphi_m y_{0m}'', \tag{7}$$

$$\sum_{m=1}^{n} \varphi_m' y_{0m}'' = 0 \qquad \Longrightarrow \qquad y''' = \sum_{m=1}^{n} \varphi_m y_{0m}''', \tag{8}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-2)} = 0 \qquad \Longrightarrow \qquad y^{(n-1)} = \sum_{m=1}^{n} \varphi_m y_{0m}^{(n-1)}. \tag{9}$$

$$y^{(n)} = \sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-1)} + \sum_{m=1}^{n} \varphi_m y_{0m}^{(n)}. \tag{10}$$

Теперь подставим:

$$\begin{gathered} \sum_{k=0}^{n} a_k y^{(k)} = \sum_{k=0}^{n-1} a_k y^{(k)} + a_n y^{(n)} = \sum_{k=0}^{n-1} a_k \sum_{m=1}^{n} \varphi_m y_{0m}^{(k)} + a_n \left(\sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-1)} + \sum_{m=1}^{n} \varphi_m y_{0m}^{(n)} \right) = \\ = \left(\sum_{k=0}^{n-1} a_k \sum_{m=1}^{n} \varphi_m y_{0m}^{(k)} + a_n \sum_{m=1}^{n} \varphi_m y_{0m}^{(n)} \right) + a_n \sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-1)} = \sum_{k=0}^{n} a_k \sum_{m=1}^{n} \varphi_m y_{0m}^{(k)} + a_n \sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-1)} = \\ = \sum_{m=1}^{n} \varphi_m \sum_{k=0}^{n} a_k y_{0m}^{(k)} + a_n \sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-1)} = a_n \sum_{m=1}^{n} \varphi_m' y_{0m}^{(n-1)} = f\left(x\right) \end{gathered} \tag{11}$$

Все условия на $\varphi_m(x)$:

$$\left\{\begin{array}{l} \varphi_1' y_{01} + \ldots + \varphi_n' y_{0n} = 0, \\ \varphi_1' y_{01}' + \ldots + \varphi_n' y_{0n}' = 0, \\ \ldots \\ \varphi_1' y_{01}^{(n-2)} + \ldots + \varphi_n' y_{0n}^{(n-2)} = 0, \\ \varphi_1' y_{01}^{(n-1)} + \ldots + \varphi_n' y_{0n}^{(n-1)} = \frac{f(x)}{a_n}; \end{array}\right. \tag{12}$$

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$$y'' + 3y' + 2y = \frac{1}{e^x + 1}. \tag{13}$$

$$y_0'' + 3y_0' + 2y_0 = 0, \tag{14}$$

характеристическое:

$$\lambda^2 + 3\lambda + 2 = 0, \qquad \lambda_1 = -2,\ \lambda_2 = -1. \tag{15}$$

Частные решения:

$$y_{01} = e^{-2x}, \qquad y_{02} = e^{-x}; \tag{16}$$

решение неоднородного уравнения будем искать в виде

$$y = e^{-2x}\varphi_1(x) + e^{-x}\varphi_2(x). \tag{17}$$

$$\left\{\begin{array}{l} e^{-2x}\varphi_1' + e^{-x}\varphi_2' = 0, \\ -2e^{-2x}\varphi_1' - e^{-x}\varphi_2' = \frac{1}{e^x+1}. \end{array}\right. \tag{18}$$

$$-e^{-2x}\varphi_1' = \frac{1}{e^x+1}, \qquad \varphi_1' = -\frac{e^{2x}}{e^x+1}, \tag{19}$$

$$e^{-x}\varphi_2' = -e^{-2x}\varphi_1', \qquad \varphi_2' = -e^{-x}\varphi_1' = \frac{e^x}{e^x+1}. \tag{20}$$

$$\varphi_1 = -\int \frac{e^{2x}}{e^x+1}dx = -\int \frac{e^x(e^x+1-1)}{e^x+1}dx = -\int \left(e^x - \frac{e^x}{e^x+1}\right)dx = -e^x + \ln(e^x+1) + C_1, \tag{21}$$

$$\varphi_2 = \int \frac{e^x}{e^x+1}dx = \ln(e^x+1) + C_2; \tag{22}$$

$$y = e^{-2x}\left[-e^x + \ln(e^x+1) + C_1\right] + e^{-x}\left[\ln(e^x+1) + C_2\right]. \tag{23}$$

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Уравнения Эйлера:

$$\sum_{k=0}^{n} a_k (bx+c)^k y^{(k)} = f(x) \tag{24}$$

$bx + c = e^t$.

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$$(x-2)^2 y'' - 3(x-2)y' + 4y = x. \tag{25}$$

Замена:

$$x - 2 = e^t, \qquad 1 = e^t t_x', \qquad t_x' = e^{-t}; \tag{26}$$

$$y' = \dot{y} t'_x = e^{-t}\dot{y}, \qquad y'' = \frac{dy'}{dt} t'_x = e^{-t}\frac{d}{dt}\left(e^{-t}\dot{y}\right) = e^{-t}\left(-e^{-t}\dot{y} + e^{-t}\ddot{y}\right) = e^{-2t}\left(\ddot{y} - \dot{y}\right). \tag{27}$$

Подставляем в уравнение (25):

$$(x-2)^2 e^{-2t}\left(\ddot{y} - \dot{y}\right) - 3e^{-t}(x-2)\,\dot{y} + 4y = e^t + 2, \tag{28}$$

$$(\ddot{y} - \dot{y}) - 3\dot{y} + 4y = e^t + 2, \tag{29}$$

$$\ddot{y} - 4\dot{y} + 4y = e^t + 2, \tag{30}$$

$$\ddot{y}_0 - 4\dot{y}_0 + 4y_0 = 0. \tag{31}$$

$$\lambda^2 - 4\lambda + 4 = 0, \qquad \lambda = 2,\ k = 2, \tag{32}$$

$$y_{01} = e^{2t}, y_{02} = te^{2t},$$

$$y_0 = C_1 e^{2t} + C_2 t e^{2t}. \tag{33}$$

$$\ddot{y}_1 - 4\dot{y}_1 + 4y_1 = e^t \tag{34}$$

$$y_1 = Ae^t \tag{35}$$

$$Ae^t - 4Ae^t + 4Ae^t = e^t \tag{36}$$

$$A = 1, \qquad y_1 = e^t \tag{37}$$

$$\ddot{y}_2 - 4\dot{y}_2 + 4y_2 = 2 \tag{38}$$

$$y_2 = B \tag{39}$$

$$4B = 2, \qquad B = \frac{1}{2}, \qquad y_2 = \frac{1}{2}. \tag{40}$$

$$y = y_0 + y_1 + y_2 = C_1 e^{2t} + C_2 t e^{2t} + e^t + \frac{1}{2} = \left(C_1 + C_2 t\right) e^{2t} + e^t + \frac{1}{2}. \tag{41}$$

$e^t = x - 2$

$$y = \left[C_1 + C_2 \ln(x-2)\right](x-2)^2 + x - \frac{3}{2}. \tag{42}$$

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