

1 Условия задачи

$$\Delta U = 0 \quad (1)$$

$$U|_{\rho=l} = \begin{cases} U_0, & 0 \leq \theta \leq \frac{\pi}{2}, \\ -U_0, & \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \quad (2)$$

2 Переход в сферические координаты

Переведём лапласиан в сферическую систему координат. В декартовой и цилиндрической системе

$$\Delta U = U_{xx} + U_{yy} + U_{zz} = U_{rr} + \frac{1}{r^2} U_{\varphi\varphi} + \frac{1}{r} U_r + U_{zz} = \frac{1}{r} \frac{\partial}{\partial r} (rU_r) + \frac{1}{r^2} U_{\varphi\varphi} + U_{zz}. \quad (3)$$

Переход между цилиндрической и сферической системами координат

$$\begin{cases} r = \rho \sin \theta \\ \varphi = \varphi \\ z = \rho \cos \theta \end{cases} \quad (4)$$

продифференцируем по каждой из цилиндрических координат:

$$\begin{cases} 1 = \rho'_r \sin \theta + \rho \cos \theta \theta'_r \\ 0 = \varphi'_r \\ 0 = \rho'_r \cos \theta - \rho \sin \theta \theta'_r \end{cases} \quad \begin{cases} 0 = \rho'_\varphi \sin \theta + \rho \cos \theta \theta'_\varphi \\ 1 = \varphi'_\varphi \\ 0 = \rho'_\varphi \cos \theta - \rho \sin \theta \theta'_\varphi \end{cases} \quad \begin{cases} 0 = \rho'_z \sin \theta + \rho \cos \theta \theta'_z \\ 0 = \varphi'_z \\ 1 = \rho'_z \cos \theta - \rho \sin \theta \theta'_z \end{cases} \quad (5)$$

Отсюда получим

$$\varphi'_r = \varphi'_z = 0, \quad \rho'_\varphi = \theta'_\varphi = 0; \quad (6)$$

$$\rho'_r = \sin \theta, \quad \theta'_r = \frac{\cos \theta}{\rho}; \quad (7)$$

$$\rho'_z = \cos \theta, \quad \theta'_z = -\frac{\sin \theta}{\rho}. \quad (8)$$

Преобразуем по одному слагаемые ΔU в цилиндрической системе. Начнём с $\frac{1}{r} \frac{\partial}{\partial r} (rU_r)$.

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r} \frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial r} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial r} \frac{\partial}{\partial \varphi} = \sin \theta \frac{\partial}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta}, \quad (9)$$

$$U_r = \sin \theta U_\rho + \frac{\cos \theta}{\rho} U_\theta, \quad (10)$$

$$rU_r = \rho \sin^2 \theta U_\rho + \frac{1}{2} \sin(2\theta) U_\theta, \quad (11)$$

$$\begin{aligned} \frac{\partial}{\partial r} (rU_r) &= \left(\sin \theta \frac{\partial}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta} \right) \left(\rho \sin^2 \theta U_\rho + \frac{1}{2} \sin(2\theta) U_\theta \right) = \\ &= \sin^3 \theta \frac{\partial}{\partial \rho} \rho U_\rho + \cos \theta \frac{\partial}{\partial \theta} \sin^2 \theta U_\rho + \frac{1}{2} \sin(2\theta) \sin \theta \frac{\partial}{\partial \rho} U_\theta + \frac{\cos \theta}{2\rho} \frac{\partial}{\partial \theta} \sin(2\theta) U_\theta, \end{aligned} \quad (12)$$

$$\begin{aligned}
\frac{1}{r} \frac{\partial}{\partial r} (rU_r) &= \frac{1}{\rho} \sin^2 \theta \frac{\partial}{\partial \rho} \rho U_\rho + \frac{\cos \theta}{\rho \sin \theta} \frac{\partial}{\partial \theta} \sin^2 \theta U_\rho + \frac{1}{2\rho} \sin(2\theta) \frac{\partial}{\partial \rho} U_\theta + \frac{\cos \theta}{2\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \sin(2\theta) U_\theta = \\
&= \frac{1}{\rho} \sin^2 \theta (\rho U_{\rho\rho} + U_\rho) + \frac{\cos \theta}{\rho \sin \theta} (\sin(2\theta) U_\rho + \sin^2 \theta U_{\rho\theta}) + \frac{1}{2\rho} \sin(2\theta) U_{\theta\rho} + \frac{\cos \theta}{2\rho^2 \sin \theta} (2 \cos(2\theta) U_\theta + \sin(2\theta) U_{\theta\theta}) = \quad (13) \\
&= \left(\sin^2 \theta U_{\rho\rho} + \frac{1}{\rho} \sin^2 \theta U_\rho \right) + \left(\frac{2 \cos^2 \theta}{\rho} U_\rho + \frac{\sin 2\theta}{2\rho} U_{\rho\theta} \right) + \frac{\sin(2\theta)}{2\rho} U_{\theta\rho} + \left(\frac{\cos \theta \cos(2\theta)}{\rho^2 \sin \theta} U_\theta + \frac{\cos^2 \theta}{\rho^2} U_{\theta\theta} \right) = \\
&= \sin^2 \theta U_{\rho\rho} + \frac{\sin 2\theta}{\rho} U_{\rho\theta} + \frac{\cos^2 \theta}{\rho^2} U_{\theta\theta} + \frac{1 + \cos^2 \theta}{\rho} U_\rho + \frac{\cos \theta \cos(2\theta)}{\rho^2 \sin \theta} U_\theta.
\end{aligned}$$

Теперь U_{zz} :

$$\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\partial \rho}{\partial z} \frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \varphi} = \cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta}, \quad (14) \\
U_{zz} &= \frac{\partial}{\partial z} U_z = \left(\cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta} \right) \left(\cos \theta U_\rho - \frac{\sin \theta}{\rho} U_\theta \right) = \\
&= \cos^2 \theta \frac{\partial}{\partial \rho} U_\rho - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta} \cos \theta U_\rho - \frac{1}{2} \sin(2\theta) \frac{\partial}{\partial \rho} \frac{1}{\rho} U_\theta + \frac{\sin \theta}{\rho^2} \frac{\partial}{\partial \theta} \sin \theta U_\theta = \\
&= \cos^2 \theta U_{\rho\rho} - \frac{\sin \theta}{\rho} (-\sin \theta U_\rho + \cos \theta U_{\rho\theta}) - \frac{1}{2} \sin(2\theta) \left(-\frac{1}{\rho^2} U_\theta + \frac{1}{\rho} U_{\theta\rho} \right) + \frac{\sin \theta}{\rho^2} (\cos \theta U_\theta + \sin \theta U_{\theta\theta}) = \quad (15) \\
&= \cos^2 \theta U_{\rho\rho} + \frac{\sin^2 \theta}{\rho} U_\rho - \frac{\sin(2\theta)}{2\rho} U_{\rho\theta} + \frac{1}{2\rho^2} \sin(2\theta) U_\theta - \frac{1}{2\rho} \sin(2\theta) U_{\theta\rho} + \frac{\sin(2\theta)}{2\rho^2} U_\theta + \frac{\sin^2 \theta}{\rho^2} U_{\theta\theta} = \\
&= \cos^2 \theta U_{\rho\rho} - \frac{\sin(2\theta)}{\rho} U_{\rho\theta} + \frac{\sin^2 \theta}{\rho^2} U_{\theta\theta} + \frac{\sin^2 \theta}{\rho} U_\rho + \frac{\sin(2\theta)}{\rho^2} U_\theta.
\end{aligned}$$

Упростим сумму

$$\begin{aligned}
\frac{1}{r} \frac{\partial}{\partial r} (rU_r) + U_{zz} &= \sin^2 \theta U_{\rho\rho} + \frac{\sin 2\theta}{\rho} U_{\rho\theta} + \frac{\cos^2 \theta}{\rho^2} U_{\theta\theta} + \frac{1 + \cos^2 \theta}{\rho} U_\rho + \frac{\cos \theta \cos(2\theta)}{\rho^2 \sin \theta} U_\theta + \\
&+ \cos^2 \theta U_{\rho\rho} - \frac{\sin(2\theta)}{\rho} U_{\rho\theta} + \frac{\sin^2 \theta}{\rho^2} U_{\theta\theta} + \frac{\sin^2 \theta}{\rho} U_\rho + \frac{\sin(2\theta)}{\rho^2} U_\theta = \\
&= U_{\rho\rho} + \frac{1}{\rho^2} U_{\theta\theta} + \frac{2}{\rho} U_\rho + \left(\frac{\cos(2\theta)}{\sin \theta} + 2 \sin(\theta) \right) \cos \theta \frac{U_\theta}{\rho^2} = \quad (16) \\
&= U_{\rho\rho} + \frac{1}{\rho^2} U_{\theta\theta} + \frac{2}{\rho} U_\rho + \frac{\cos \theta}{\sin \theta} \frac{U_\theta}{\rho^2},
\end{aligned}$$

и, наконец, вычислим весь лапласиан:

$$\Delta U = \frac{1}{r} \frac{\partial}{\partial r} (rU_r) + \frac{1}{r^2} U_{\varphi\varphi} + U_{zz} = U_{\rho\rho} + \frac{1}{\rho^2} U_{\theta\theta} + \frac{1}{\rho^2 \sin^2 \theta} U_{\varphi\varphi} + \frac{2}{\rho} U_\rho + \frac{\cos \theta}{\sin \theta} \frac{U_\theta}{\rho^2} \quad (17)$$

3 Разделение переменных

Будем искать решения уравнения Лапласа, которое примет вид

$$U_{\rho\rho} + \frac{1}{\rho^2}U_{\theta\theta} + \frac{1}{\rho^2 \sin^2 \theta}U_{\varphi\varphi} + \frac{2}{\rho}U_{\rho} + \frac{\cos \theta}{\sin \theta} \frac{U_{\theta}}{\rho^2} = 0, \quad (18)$$

в виде произведений

$$U(\rho, \theta, \varphi) = R(\rho) \Theta(\theta) \Phi(\varphi). \quad (19)$$

Как всегда,

$$\Phi(\varphi) = \Phi(\varphi + 2\pi). \quad (20)$$

Начнём

$$R''\Theta\Phi + \frac{1}{\rho^2}R\Theta''\Phi + \frac{1}{\rho^2 \sin^2 \theta}R\Theta\Phi'' + \frac{2}{\rho}R'\Theta\Phi + \frac{\cos \theta}{\sin \theta} \frac{R\Theta'\Phi}{\rho^2} = 0 \left| \cdot \frac{\rho^2 \sin^2 \theta}{R\Theta\Phi} \right. \quad (21)$$

$$\frac{R''}{R}\rho^2 \sin^2 \theta + \frac{\Theta''}{\Theta} \sin^2 \theta + \frac{\Phi''}{\Phi} + 2 \sin^2 \theta \rho \frac{R'}{R} + \sin \theta \cos \theta \frac{\Theta'}{\Theta} = 0 \quad (22)$$

$$\frac{R''}{R}\rho^2 \sin^2 \theta + \frac{\Theta''}{\Theta} \sin^2 \theta + 2 \sin^2 \theta \rho \frac{R'}{R} + \sin \theta \cos \theta \frac{\Theta'}{\Theta} = -\frac{\Phi''}{\Phi} = \mu, \quad \mu = const. \quad (23)$$

3.1 Долготная часть

$$-\frac{\Phi''}{\Phi} = \mu \quad (24)$$

из (24) и (20), как в задаче 60 получаем

$$\mu = n^2, \quad (25)$$

$$\Phi_n = C_n^1 \cos(n\varphi) + C_n^2 \sin(n\varphi). \quad (26)$$

$$\frac{R''}{R}\rho^2 \sin^2 \theta + \frac{\Theta''}{\Theta} \sin^2 \theta + 2 \sin^2 \theta \rho \frac{R'}{R} + \sin \theta \cos \theta \frac{\Theta'}{\Theta} = n^2 \left| \cdot \frac{1}{\sin^2 \theta} \right. \quad (27)$$

$$\frac{R''}{R}\rho^2 + 2\rho \frac{R'}{R} + \frac{\Theta''}{\Theta} + \frac{\cos \theta}{\sin \theta} \frac{\Theta'}{\Theta} = \frac{n^2}{\sin^2 \theta} \quad (28)$$

$$\frac{\Theta''}{\Theta} + \frac{\cos \theta}{\sin \theta} \frac{\Theta'}{\Theta} - \frac{n^2}{\sin^2 \theta} = -\frac{R''}{R}\rho^2 - 2\rho \frac{R'}{R} = \varkappa, \quad \varkappa = const \quad (29)$$

3.2 Широтная часть

$$\frac{\Theta''}{\Theta} + \frac{\cos \theta}{\sin \theta} \frac{\Theta'}{\Theta} - \frac{n^2}{\sin^2 \theta} = \varkappa \left| \cdot \Theta \right. \quad (30)$$

$$\Theta'' + \frac{\cos \theta}{\sin \theta} \Theta' + \left(-\varkappa - \frac{n^2}{\sin^2 \theta} \right) \Theta = 0 \quad (31)$$

Произведём замену

$$\cos \theta = x \quad (32)$$

$$\sin^2 \theta = 1 - \cos^2 \theta = 1 - x^2, \quad \frac{dx}{d\theta} = -\sin \theta \quad (33)$$

$$\Theta' = \frac{d\Theta}{dx} \frac{dx}{d\theta} = -\sin \theta \frac{d\Theta}{dx}, \quad \Theta'' = \frac{d}{d\theta} \Theta' = -\frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{dx} \right) = \sin^2 \theta \frac{d^2 \Theta}{dx^2} - \cos \theta \frac{d\Theta}{dx} \quad (34)$$

$$\left(\sin^2 \theta \frac{d^2 \Theta}{dx^2} - \cos \theta \frac{d\Theta}{dx} \right) + \frac{\cos \theta}{\sin \theta} \left(-\sin \theta \frac{d\Theta}{dx} \right) + \left(-\varkappa - \frac{n^2}{\sin^2 \theta} \right) \Theta = 0 \quad (35)$$

$$\sin^2 \theta \frac{d^2 \Theta}{dx^2} - \cos \theta \frac{d\Theta}{dx} - \cos \theta \frac{d\Theta}{dx} + \left(-\varkappa - \frac{n^2}{\sin^2 \theta} \right) \Theta = 0 \quad (36)$$

$$(1 - x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + \left(-\varkappa - \frac{n^2}{1 - x^2} \right) \Theta = 0 \quad (37)$$

Это уравнение имеет нетривиальные решения (что следует из теории) только при

$$-\varkappa = m(m+1), \quad (38)$$

которые записываются так:

$$\Theta = C_{nm}^3 P_m^n(x) = C_{nm}^3 P_m^n(\cos \theta). \quad (39)$$

3.3 Радиальная часть

$$-\frac{R''}{R} \rho^2 - 2\rho \frac{R'}{R} = \varkappa = -m(m+1), \quad (40)$$

$$R'' \rho^2 + 2\rho R' - m(m+1)R = 0 \quad (41)$$

уравнение Эйлера; $\rho = e^x$, $x = \ln \rho$

$$R' = \frac{1}{\rho} \frac{dR}{dx}, \quad (42)$$

$$R'' = \frac{1}{\rho^2} \left(\frac{d^2 R}{dx^2} - \frac{dR}{dx} \right), \quad (43)$$

$$\frac{d^2 R}{dx^2} + \frac{dR}{dx} - m(m+1)R = 0 \quad (44)$$

Характеристическое уравнение:

$$\lambda^2 + \lambda - m(m+1) = 0 \quad (45)$$

$$\lambda = m, \quad \lambda = -m-1 \quad (46)$$

$$R = C_m^4 e^{mx} + C_m^5 e^{-(m+1)x} = C_m^4 \rho^m + C_m^5 \rho^{-(m+1)} = C_m^4 \rho^m \quad (47)$$

Общее решение:

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_m^4 \rho^m C_{nm}^3 P_m^n(\cos \theta) [C_n^1 \cos(n\varphi) + C_n^2 \sin(n\varphi)] =$$

$$C_m^4 C_{nm}^3 C_n^1 = C_{mn}^6, \quad C_m^4 C_{nm}^3 C_n^2 = C_{mn}^7$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho^m P_m^n(\cos \theta) [C_{mn}^6 \cos(n\varphi) + C_{mn}^7 \sin(n\varphi)] \quad (48)$$

4 Граничные условия

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} l^m P_m^n(\cos \theta) [C_{mn}^6 \cos(n\varphi) + C_{mn}^7 \sin(n\varphi)] = \begin{cases} U_0, & 0 \leq \theta \leq \frac{\pi}{2}, \\ -U_0, & \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \quad (49)$$

Т.к. граничные условия не зависят от φ , повторив выводы из задачи 114, получим, что

$$C_{mn}^7 = 0; \quad C_{mn}^6 = 0, \quad n \neq 0. \quad (50)$$

Ненулевыми могут быть только C_{m0}^6 , и

$$U = \sum_{m=0}^{\infty} C_{m0}^6 \rho^m P_m(\cos \theta) \quad (51)$$

$$\sum_{m=0}^{\infty} C_{m0}^6 l^m P_m(\cos \theta) = \begin{cases} U_0, & 0 \leq \theta \leq \frac{\pi}{2}, \\ -U_0, & \frac{\pi}{2} \leq \theta \leq \pi. \end{cases} \quad (52)$$

Снова совершим замену (32):

$$\sum_{m=0}^{\infty} C_{m0}^6 l^m P_m(x) = f(x) = \begin{cases} U_0, & 0 \leq x \leq 1, \\ -U_0, & -1 \leq x \leq 0. \end{cases} \quad (53)$$

Чтобы найти C_{m0}^6 , воспользуемся свойством

$$\int_{-1}^1 P_m(x) P_k(x) dx = \delta_{mk} \frac{2}{2k+1} \quad (54)$$

$$\int_{-1}^1 \sum_{m=0}^{\infty} C_{m0}^6 l^m P_m(x) P_k(x) dx = \int_{-1}^1 f(x) P_k(x) dx \quad (55)$$

Левая часть:

$$\int_{-1}^1 \sum_{m=0}^{\infty} C_{m0}^6 l^m P_m(x) P_k(x) dx = \sum_{m=0}^{\infty} C_{m0}^6 l^m \int_{-1}^1 P_m(x) P_k(x) dx = \sum_{m=0}^{\infty} C_{m0}^6 l^m \delta_{mk} \frac{2}{2k+1} = C_{k0}^6 \frac{2l^k}{2k+1} \quad (56)$$

Правая часть:

$$\begin{aligned} \int_{-1}^1 f(x) P_k(x) dx &= \int_{-1}^0 f(x) P_k(x) dx + \int_0^1 f(x) P_k(x) dx = -U_0 \int_{-1}^0 P_k(x) dx + U_0 \int_0^1 P_k(x) dx = \\ &= U_0 \left(\int_0^1 P_k(x) dx - \int_{-1}^0 P_k(x) dx \right) \end{aligned} \quad (57)$$

По формуле Родрига

$$P_k(x) = \frac{1}{k! 2^k} \frac{d^k}{dx^k} (x^2 - 1)^k. \quad (58)$$

При $k = 0$ $P_0(x) = 1$ и

$$\int_{-1}^1 f(x) P_0(x) dx = U_0 \left(\int_0^1 P_0(x) dx - \int_{-1}^0 P_0(x) dx \right) = U_0 \left(\int_0^1 dx - \int_{-1}^0 dx \right) = U_0(1 - 1) = 0. \quad (59)$$

При $k > 0$

$$\begin{aligned} \int_{-1}^1 f(x) P_k(x) dx &= \frac{U_0}{k!2^k} \left(\int_0^1 \frac{d^k}{dx^k} (x^2 - 1)^k dx - \int_{-1}^0 \frac{d^k}{dx^k} (x^2 - 1)^k dx \right) = \frac{U_0}{k!2^k} \left(\frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_0^1 - \frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_{-1}^0 \right) = \\ &= \frac{U_0}{k!2^k} \left(- \frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_{x=0} - \frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_{x=0} \right) = - \frac{U_0}{k!2^{k-1}} \frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_{x=0} \end{aligned} \quad (60)$$

Вычислим отдельно

$$(x^2 - 1)^k = \sum_{m=0}^k C_k^m x^{2m} (-1)^{k-m}, \quad (61)$$

При $k = 2n$

$$\begin{aligned} \frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k &= \frac{d^{2n-1}}{dx^{2n-1}} (x^2 - 1)^{2n} = \frac{d^{2n-1}}{dx^{2n-1}} \sum_{m=0}^{2n} C_{2n}^m x^{2m} (-1)^{2n-m} = \\ &= \frac{d^{2n-1}}{dx^{2n-1}} \left(\sum_{m=0}^{n-1} C_{2n}^m x^{2m} (-1)^{2n-m} + \sum_{m=n}^{2n} C_{2n}^m x^{2m} (-1)^{2n-m} \right) = \sum_{m=n}^{2n} C_{2n}^m \frac{(2m)!}{(2m - 2n + 1)!} x^{2m-2n+1} (-1)^{2n-m}, \end{aligned} \quad (62)$$

$$\frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_{x=0} = 0, \quad (63)$$

$$\int_{-1}^1 f(x) P_{2n}(x) dx = - \frac{U_0}{k!2^{k-1}} \frac{d^{k-1}}{dx^{k-1}} (x^2 - 1)^k \Big|_{x=0} = 0. \quad (64)$$

При $k = 2n - 1$

$$\begin{aligned} \frac{d^{2n-2}}{dx^{2n-2}} (x^2 - 1)^{2n-1} &= \frac{d^{2n-2}}{dx^{2n-2}} \sum_{m=0}^{2n-1} C_{2n-1}^m x^{2m} (-1)^{2n-1-m} = \\ &= \frac{d^{2n-2}}{dx^{2n-2}} \left(\sum_{m=0}^{n-2} C_{2n-1}^m x^{2m} (-1)^{2n-1-m} + C_{2n-1}^{n-1} x^{2n-2} (-1)^n + \sum_{m=n}^{2n-1} C_{2n-1}^m x^{2m} (-1)^{2n-1-m} \right) = \end{aligned} \quad (65)$$

$$= C_{2n-1}^{n-1} (2n-2)! (-1)^n + \sum_{m=n}^{2n-1} C_{2n-1}^m \frac{(2m)!}{(2m - 2n + 2)!} x^{2m-2n+2} (-1)^{2n-1-m},$$

$$\frac{d^{2n-2}}{dx^{2n-2}} (x^2 - 1)^{2n-1} \Big|_{x=0} = C_{2n-1}^{n-1} (2n-2)! (-1)^n, \quad (66)$$

$$\begin{aligned}
\int_{-1}^1 f(x) P_{2n-1}(x) dx &= -\frac{U_0}{(2n-1)!2^{2n-2}} \frac{d^{2n-2}}{dx^{2n-2}} (x^2-1)^{2n-1} \Big|_{x=0} = -\frac{U_0}{(2n-1)!2^{2n-2}} C_{2n-1}^{n-1} (2n-2)! (-1)^n = \\
&= -\frac{(-1)^n U_0 (2n-2)!}{2^{2n-2} n! (n-1)!}
\end{aligned} \tag{67}$$

Итак,

$$C_{2n,0}^6 \frac{2l^{2n}}{4n+1} = 0 \implies C_{2n,0}^6 = 0; \tag{68}$$

$$C_{2n-1,0}^6 \frac{2l^{2n-1}}{4n-1} = -\frac{(-1)^n U_0 (2n-2)!}{2^{2n-2} n! (n-1)!}, \tag{69}$$

$$C_{2n-1,0}^6 = -\frac{(-1)^n U_0 (4n-1)(2n-2)!}{(2l)^{2n-1} n! (n-1)!} \tag{70}$$

По формуле (51)

$$\begin{aligned}
U &= \sum_{m=0}^{\infty} C_{m,0}^6 \rho^m P_m(\cos \theta) = \sum_{n=1}^{\infty} (C_{2n,0}^6 \rho^{2n} P_{2n}(\cos \theta) + C_{2n-1,0}^6 \rho^{2n-1} P_{2n-1}(\cos \theta)) = \\
&= -U_0 \sum_{n=1}^{\infty} \frac{(-1)^n (4n-1)(2n-2)!}{(2l)^{2n-1} n! (n-1)!} \rho^{2n-1} P_{2n-1}(\cos \theta)
\end{aligned} \tag{71}$$