1 Что уже было

$$T\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)=\alpha\in\mathbb{R},\qquad \vec{x}_\alpha\in L,\quad \omega^\beta\in L^* \tag{1}$$

$$T\left(\vec{x}_1,\ldots,a\vec{x}_\alpha+b\vec{y},\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)=aT\left(\vec{x}_1,\ldots,\vec{x}_\alpha,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)+bT\left(\vec{x}_1,\ldots,\vec{y},\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right) \tag{2}$$

$$\vec{x}_\alpha=x^i_\alpha\vec{e}_i\in L,\quad \omega^j=\omega^\beta_j\epsilon^j\in L^*,\quad \epsilon^j\left(\vec{e}_i\right)=\delta^j_i. \tag{3}$$

$$T\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)=T\left(x_1^{i_1}\vec{e}_{i_1},\ldots,x_p^{i_p}\vec{e}_{i_p},\,\omega^1_{j_1}\epsilon^{j_1},\ldots,\omega^q_{j_q}\epsilon^{j_q}\right)=$$

$$=x_1^{i_1}\ldots x_p^{i_p}\omega^1_{j_1}\ldots\omega^q_{j_q}T\left(\vec{e}_{i_1},\ldots,\vec{e}_{i_p},\,\epsilon^{j_1},\ldots,\epsilon^{j_q}\right)=x_1^{i_1}\ldots x_p^{i_p}\omega^1_{j_1}\ldots\omega^q_{j_q}T^{j_1\ldots j_q}_{i_1\ldots i_p} \tag{4}$$

$$T^{j_1\ldots j_q}_{i_1\ldots i_p}\equiv T\left(\vec{e}_{i_1},\ldots,\vec{e}_{i_p},\,\epsilon^{j_1},\ldots,\epsilon^{j_q}\right) \tag{5}$$

Преобразование компонент при смене базиса

$$x^{i'}=\left(S^{-1}\right)^{i'}_i x^i,\qquad ,\omega_{j'}=S^j_{j'}\omega_j \tag{6}$$

$$T=x_1^{i_1}\ldots x_p^{i_p}\omega^1_{j_1}\ldots\omega^q_{j_q}T^{j_1\ldots j_q}_{i_1\ldots i_p}=x_1^{i'_1}\ldots x_p^{i'_p}\omega^1_{j'_1}\ldots\omega^q_{j'_q}T^{j'_1\ldots j'_q}_{i'_1\ldots i'_p}=\left(S^{-1}\right)^{i'_1}_{i_1}x^{i_1}\ldots\left(S^{-1}\right)^{i'_p}_{i_p}x^{i_p}_pS^{j_1}_{j'_1}\omega^1_{j_1}\ldots S^{j_q}_{j'_q}\omega^q_{j_q}T^{j'_1\ldots j'_q}_{i'_1\ldots i'_p} \tag{7}$$

$$x_1^{i_1}\ldots x_p^{i_p}\omega^1_{j_1}\ldots\omega^q_{j_q}T^{j_1\ldots j_q}_{i_1\ldots i_p}-\left(S^{-1}\right)^{i'_1}_{i_1}x^{i_1}\ldots\left(S^{-1}\right)^{i'_p}_{i_p}x^{i_p}_pS^{j_1}_{j'_1}\omega^1_{j_1}\ldots S^{j_q}_{j'_q}\omega^q_{j_q}T^{j'_1\ldots j'_q}_{i'_1\ldots i'_p}=0 \tag{8}$$

$$x_1^{i_1}\ldots x_p^{i_p}\omega^1_{j_1}\ldots\omega^q_{j_q}\left(T^{j_1\ldots j_q}_{i_1\ldots i_p}-\left(S^{-1}\right)^{i'_1}_{i_1}\ldots\left(S^{-1}\right)^{i'_p}_{i_p}S^{j_1}_{j'_1}\ldots S^{j_q}_{j'_q}T^{j'_1\ldots j'_q}_{i'_1\ldots i'_p}\right)=0 \tag{9}$$

$\forall\ x_1^{i_1}\ldots x_p^{i_p}\omega^1_{j_1}\ldots\omega^q_{j_q}$:

$$T^{j_1\ldots j_q}_{i_1\ldots i_p}=\left(S^{-1}\right)^{i'_1}_{i_1}\ldots\left(S^{-1}\right)^{i'_p}_{i_p}S^{j_1}_{j'_1}\ldots S^{j_q}_{j'_q}T^{j'_1\ldots j'_q}_{i'_1\ldots i'_p} \tag{10}$$

и обратно:

$$T^{j'_1\ldots j'_q}_{i'_1\ldots i'_p}=S^{i_1}_{i'_1}\ldots S^{i_p}_{i'_p}\left(S^{-1}\right)^{j'_1}_{j_1}\ldots\left(S^{-1}\right)^{j'_q}_{j_q}T^{j_1\ldots j_q}_{i_1\ldots i_p} \tag{11}$$

2 Действия с тензорами

2.1 Умножение на число

$$S\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)=aT\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right) \tag{12}$$

$$S^{j_1\ldots j_q}_{i_1\ldots i_p}=aT^{j_1\ldots j_q}_{i_1\ldots i_p} \tag{13}$$

2.2 Умножение

$$T_1\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)=\alpha,\qquad T_2\left(\vec{y}_1,\ldots,\vec{y}_s,\nu^1,\ldots,\nu^t\right)=\beta \tag{14}$$

$$T_1\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)T_2\left(\vec{y}_1,\ldots,\vec{y}_s,\nu^1,\ldots,\nu^t\right)=\alpha\beta \tag{15}$$

$$T_1\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)T_2\left(\vec{y}_1,\ldots,a\vec{y}_k+b\vec{z},\ldots,\vec{y}_s,\nu^1,\ldots,\nu^t\right)= \tag{16}$$

$$=aT_1\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)T_2\left(\vec{y}_1,\ldots,\vec{y}_k,\ldots,\vec{y}_s,\nu^1,\ldots,\nu^t\right)+bT_1\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)T_2\left(\vec{y}_1,\ldots,\vec{z},\ldots,\vec{y}_s,\nu^1,\ldots,\nu^t\right)$$

$$T_1\left(\vec{x}_1,\ldots,\vec{x}_p,\omega^1,\ldots,\omega^q\right)T_2\left(\vec{y}_1,\ldots,\vec{y}_s,\nu^1,\ldots,\nu^t\right)\equiv S\left(\vec{x}_1,\ldots,\vec{x}_p,\vec{y}_1,\ldots,\vec{y}_s,\omega^1,\ldots,\omega^q,\nu^1,\ldots,\nu^t\right) \tag{17}$$

$$S^{j_1\ldots j_{q+t}}_{i_1\ldots i_{p+s}}\equiv S\left(\vec{e}_{i_1},\ldots,\vec{e}_{i_p},\vec{e}_{i_{p+1}},\ldots,\vec{e}_{i_{p+s}},\epsilon_{j_1},\ldots,\epsilon_{j_q},\epsilon_{j_{q+1}},\ldots,\epsilon_{j_{q+t}}\right)=\left(T_1\right)^{j_1\ldots j_q}_{i_1\ldots i_p}\left(T_2\right)^{j_1\ldots j_t}_{i_1\ldots i_s} \tag{18}$$

2.3 Сложение

$$T_1+T_2=\alpha+\beta, \tag{19}$$

$$T_1\left(\vec{x}_1,\ldots,\omega^q\right)+T_2\left(\ldots,a\vec{y}_k+b\vec{z},\ldots\right)=T_1+aT_2\left(\ldots,\vec{y}_k,\ldots\right)+bT_2\left(\ldots,\vec{z},\ldots\right)\neq a\left(T_1+T_2\right)+b\left(T_1+T_2\right) \tag{20}$$

$$\left(T_1+T_2\right)\left(\vec{x}_1,\ldots,\omega^q\right)=T_1\left(\vec{x}_1,\ldots,\omega^q\right)+T_2\left(\vec{x}_1,\ldots,\omega^q\right)=S\left(\vec{x}_1,\ldots,\omega^q\right) \tag{21}$$

$$S^{j_1\ldots j_q}_{i_1\ldots i_p}=\left(T_1\right)^{j_1\ldots j_q}_{i_1\ldots i_p}+\left(T_2\right)^{j_1\ldots j_t}_{i_1\ldots i_s} \tag{22}$$

2.4 Перестановка индексов

$$T^{j_1\ldots k\ldots m\ldots j_q}_{i_1\ldots i_p}\longrightarrow S^{j_1\ldots k\ldots m\ldots j_q}_{i_1\ldots i_p}=T^{j_1\ldots m\ldots k\ldots j_q}_{i_1\ldots i_p} \tag{23}$$

(аналогично снизу)

2.4.1 Симметризация

$$T^{j_1\dots(j_k\dots j_q)}_{i_1\dots i_p} = \frac{1}{(q-k+1)!}\sigma^{j_k\dots j_q}_{j'_k\dots j'_q} T^{j_1\dots j'_k\dots j'_q}_{i_1\dots i_p} \tag{24}$$

$$\sigma^{j_k\dots j_q}_{j'_k\dots j'_q} = \begin{cases} 1, & \{j_k,\dots,j_q\} = \{j'_k,\dots,j'_q\} \\ 0 \end{cases} \tag{25}$$

$$A^{(ik)} = \frac{1}{2}\left(A^{ik} + A^{ki}\right) \tag{26}$$

$$B^{(ijk)} = \frac{1}{6}\left(B^{ijk} + B^{ikj} + B^{kij} + B^{kji} + B^{jki} + B^{jik}\right) \tag{27}$$

2.4.2 Антисимметризация

$$T^{j_1\dots[j_k\dots j_q]}_{i_1\dots i_p} = \frac{1}{(q-k+1)!}\delta^{j_k\dots j_q}_{j'_k\dots j'_q} T^{j_1\dots j'_k\dots j'_q}_{i_1\dots i_p} \tag{28}$$

$$A^{[ik]} = \frac{1}{2}\left(A^{ik} - A^{ki}\right) \tag{29}$$

$$B^{[ijk]} = \frac{1}{6}\left(B^{ijk} - B^{ikj} + B^{kij} - B^{kji} + B^{jki} - B^{jik}\right) \tag{30}$$

2.5 Свёртка индексов

$$T\left(\vec{x}_1,\dots,\vec{x}_p,\omega^1,\dots,\omega^q\right) \longrightarrow T^{j_1\dots j_{k-1}j_kj_{k+1}\dots j_q}_{i_1\dots i_{m-1}i_mi_{m+1}\dots i_p} \longrightarrow \left(\sum_{r=1}^{n}\right) T^{j_1\dots j_{k-1}rj_{k+1}\dots j_q}_{i_1\dots i_{m-1}ri_{m+1}\dots i_p} = S^{j_1\dots j_{k-1}j_{k+1}\dots j_q}_{i_1\dots i_{m-1}i_{m+1}\dots i_p} \longrightarrow S\left(\vec{x}_1,\dots,\vec{x}_{p-1},\omega^1,\dots,\omega^{q-1}\right) \tag{31}$$

3 Некоторые тензоры

3.1 Тензор Кронекера ($n = 4$)

$$\delta^i_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

$$\begin{aligned} T^{j_1\dots j_k\dots j_q}_{i_1\dots i_p}\delta^m_{j_k} &= T^{j_1\dots 1\dots j_q}_{i_1\dots i_p}\delta^m_1 + \dots + T^{j_1\dots m\dots j_q}_{i_1\dots i_p}\delta^m_m + \dots + T^{j_1\dots n\dots j_q}_{i_1\dots i_p}\delta^m_n = \\ &= 0 + \dots + T^{j_1\dots m\dots j_q}_{i_1\dots i_p}\cdot 1 + \dots + 0 = T^{j_1\dots m\dots j_q}_{i_1\dots i_p} \end{aligned} \tag{33}$$

$$\delta^{i'}_{j'} = S^j_{j'}\left(S^{-1}\right)^{i'}_i\delta^i_j = S^j_{j'}\left(S^{-1}\right)^{i'}_j = S^j_{j'}\left(S^{-1}\right)^{i'}_j = \left(S^{-1}\right)^{i'}_j S^j_{j'} = \left(S^{-1}\cdot S\right)^{i'}_{j'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

3.2 Метрический тензор

$$g_{ik} = g_{ki}, \qquad \det\|g_{ik}\| \neq 0 \tag{35}$$

Скалярное произведение векторов:

$$(X\cdot Y) = G\,(X,Y) = g_{ik}x^iy^k \tag{36}$$

$$g_{ik}g^{kj} = \delta^j_i \tag{37}$$

$$(\varphi\cdot\omega) = \tilde{G}\,(\varphi,\omega) = g^{kj}\varphi_k\omega_j$$

Тензор Минковского:

$$\eta_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \eta^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{38}$$

$$\eta^{ik}\eta_{kj} = \delta^i_j \tag{39}$$

Поднятие и опускание индексов

$$T^{j_1\dots j_{k-1}j_kj_{k+1}\dots j_q}_{i_1\dots i_p}g_{j_km} = T^{j_1\dots j_{k-1}j_{k+1}\dots j_q}_{i_1\dots i_p} \tag{40}$$

$$A^{\cdot\cdot m}_{ij\cdot} = A_{ijk}g^{km}, \quad A^{\cdot m\cdot}_{i\cdot k} = A_{ijk}g^{jm}, \quad R_{tjkl} = R^i_{\cdot jkl}g_{it} \tag{41}$$

4 т-р Леви-Чивита ($n = 3$)

$$e^{ijk} = e^{[ijk]} \tag{42}$$

абсолютно антисимметричный объект

$$e^{ijk} = -e^{ikj} = e^{kij} = -e^{kji} = e^{jki} = -e^{jik} \tag{43}$$

$$e^{123} = 1 \tag{44}$$

$$e^{ijj} = \frac{1}{2}\left(e^{ijj} + e^{ijj}\right) = \frac{1}{2}\left(e^{ijj} - e^{ijj}\right) = 0 \tag{45}$$

$$e_{123} = 1 \tag{46}$$

$$e^{ijk}e_{pqr} = \delta^i_p\delta^j_q\delta^k_r - \delta^i_p\delta^j_r\delta^k_q + \delta^i_r\delta^j_p\delta^k_q - \delta^i_r\delta^j_q\delta^k_p + \delta^i_q\delta^j_r\delta^k_p - \delta^i_q\delta^j_p\delta^k_r \tag{47}$$

$$e^{ijk} = e^{ijk}e_{123} = \delta^i_1\delta^j_2\delta^k_3 - \delta^i_1\delta^j_3\delta^k_2 + \delta^i_3\delta^j_1\delta^k_2 - \delta^i_3\delta^j_2\delta^k_1 + \delta^i_2\delta^j_3\delta^k_1 - \delta^i_2\delta^j_1\delta^k_3 \tag{48}$$

$$\begin{aligned} B_{pqr} = e^{ijk}A_{ip}A_{jq}A_{kr} = \left(\delta^i_1\delta^j_2\delta^k_3 - \delta^i_1\delta^j_3\delta^k_2 + \delta^i_3\delta^j_1\delta^k_2 - \delta^i_3\delta^j_2\delta^k_1 + \delta^i_2\delta^j_3\delta^k_1 - \delta^i_2\delta^j_1\delta^k_3\right) A_{ip}A_{jq}A_{kr} = \\ = A_{1p}A_{2q}A_{3r} - A_{1p}A_{3q}A_{2r} + A_{3p}A_{1q}A_{2r} - A_{3p}A_{2q}A_{1r} + A_{2p}A_{3q}A_{1r} - A_{2p}A_{1q}A_{3r} = \\ = \left(A_{1p}A_{2q}A_{3r} - A_{2p}A_{1q}A_{3r}\right) + \left(A_{2p}A_{3q}A_{1r} - A_{1p}A_{3q}A_{2r}\right) + \left(A_{3p}A_{1q}A_{2r} - A_{3p}A_{2q}A_{1r}\right) = \\ = \left(A_{1p}A_{2q}A_{3r} - A_{1p}A_{3q}A_{2r}\right) + \left(A_{3p}A_{1q}A_{2r} - A_{2p}A_{1q}A_{3r}\right) + \left(A_{2p}A_{3q}A_{1r} - A_{3p}A_{2q}A_{1r}\right) \end{aligned} \tag{49}$$

$$B_{pqr} = B_{[pq]r} = B_{p[qr]} \tag{50}$$

$$B_{pqr} = -B_{prq} = B_{rpq} = -B_{rqp} \tag{51}$$

$$B_{pqr} = B_{[pqr]} \tag{52}$$

$$B_{pqr} = B_{123}e_{pqr} \tag{53}$$

$$\begin{aligned} B_{123} = A_{11}A_{22}A_{33} - A_{11}A_{32}A_{23} + A_{31}A_{12}A_{23} - A_{31}A_{22}A_{13} + A_{21}A_{32}A_{13} - A_{21}A_{12}A_{33} = \\ = A_{11}\left(A_{22}A_{33} - A_{32}A_{23}\right) - A_{12}\left(A_{21}A_{33} - A_{31}A_{23}\right) + A_{13}\left(A_{21}A_{32} - A_{31}A_{22}\right) = \\ = A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} = \\ = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \end{aligned} \tag{54}$$

$$e^{ijk}A_{ip}A_{jq}A_{kr} = \det(A)\, e_{pqr}. \tag{55}$$

Аналогично,

$$e_{ijk}A^{ip}A^{jq}A^{kr} = \det(A)\, e^{pqr}. \tag{56}$$

Евклидово пространство $g_{ik} = \delta_{ik}$:

$$\delta_{i'k'} = S^i_{i'}S^k_{k'}\delta_{ik} = S^i_{i'}\delta_{ik}S^k_{k'} = S^i_{i'}\left(E \cdot S\right)_{ik'} = \left(S^T \cdot E \cdot S\right)_{i'k'} \tag{57}$$

для орт.

$$E = S^T E S = S^T S \tag{58}$$

$$\det(E) = 1 = \det\left(S^T S\right) = \det\left(S^T\right)\det(S) = \det(S)^2 \tag{59}$$

$$\det(S) = \pm 1 \tag{60}$$

Для вращений

$$\det(S) = 1, \tag{61}$$

но для отражений

$$\det(S) = -1. \tag{62}$$

Абсолютно антисимметричный объект при вращениях

$$e'_{i'j'k'} = e_{ijk}S^i_{i'}S^j_{j'}S^k_{k'} = \det(S)\, e_{i'j'k'} = e_{i'j'k'} \tag{63}$$

и отражениях

$$e'_{i'j'k'} = \det(S)\, e_{i'j'k'} = -e_{i'j'k'} \tag{64}$$

псевдотензор

5 Задачи

5.1 220

$$a_i^i = a_1^1 + a_2^2 + a_3^3 = 2 - 5 + 4 = 1 \tag{65}$$

$$a_{j'}^{i'} = \left(S^{-1}\right)_i^{i'} S_{j'}^j a_j^i \tag{66}$$

$$a_{i'}^{i'} = \left(S^{-1}\right)_i^{i'} S_{i'}^j a_j^i = S_{i'}^j \left(S^{-1}\right)_i^{i'} a_j^i = \left(S \cdot S^{-1}\right)_i^j a_j^i = \delta_i^j a_j^i = a_i^i \tag{67}$$

5.2 221 а)

$$a_{ij} x^j = a_{i1} x^1 + a_{i2} x^2 + a_{i3} x^3 \tag{68}$$

$$a_{1j} x^j = a_{11} x^1 + a_{12} x^2 + a_{13} x^3 = 2 \cdot 2 + 0 \cdot 1 + 3 \cdot 4 = 16 \tag{69}$$

$$a_{2j} x^j = a_{21} x^1 + a_{22} x^2 + a_{23} x^3 = 3 \cdot 2 + 1 \cdot 1 + 2 \cdot 4 = 15 \tag{70}$$

$$a_{3j} x^j = a_{31} x^1 + a_{32} x^2 + a_{33} x^3 = 4 \cdot 2 + 5 \cdot 1 + 7 \cdot 4 = 41 \tag{71}$$

$$a_{ij} x^j = (16, 15, 41) \tag{72}$$

Шпур:

$$\mathrm{Sp}\left(a^{ij}\right) \equiv g_{ij} a^{ij} = a_i^i = a_1^1 + \ldots a_n^n \tag{73}$$

5.3 224

$$a_{ijk} = a_{jik} = -a_{ikj} \tag{74}$$

$$a_{ijk} = -a_{ikj} = -a_{kij} \tag{75}$$

$$a_{ijk} = -a_{kij} = a_{jki} = -a_{ijk} \tag{76}$$

$$a_{ijk} = \frac{1}{2}\left(a_{ijk} + a_{ijk}\right) = \frac{1}{2}\left(a_{ijk} - a_{ijk}\right) = 0 \tag{77}$$

5.4 226

$$a_{ij} x^j = \alpha x_i = \alpha g_{ij} x^j \tag{78}$$

$$a_{ij} x^j - \alpha g_{ij} x^j = 0 \tag{79}$$

$$\left(a_{ij} - \alpha g_{ij}\right) x^j = 0 \tag{80}$$

$\forall x^j$:

$$a_{ij} - \alpha g_{ij} = 0 \tag{81}$$

$$a_{ij} = \alpha g_{ij} \tag{82}$$