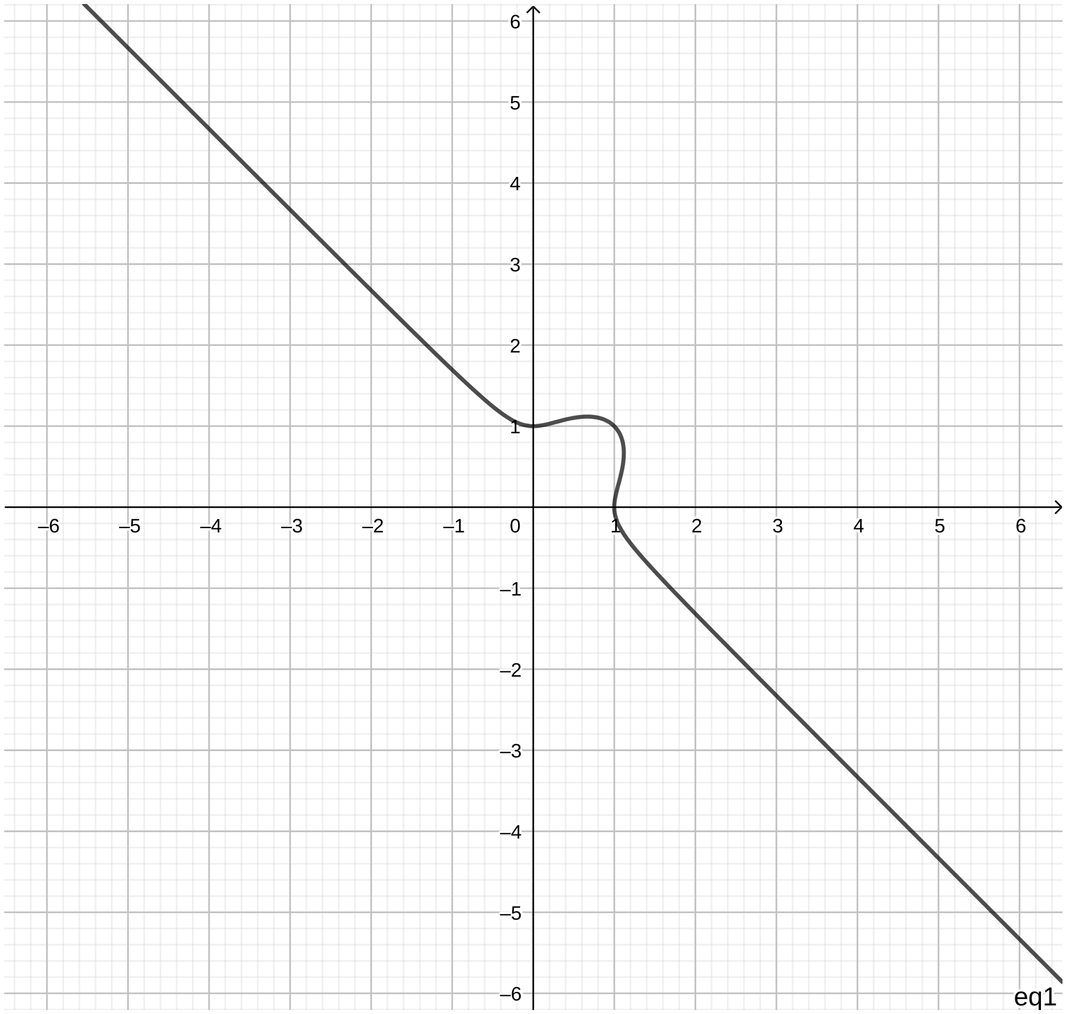
Вводя обобщённые полярные координаты, найти площадь, ограниченную кривыми:

$$\frac{x^3}{a^3}+\frac{y^3}{b^3}=\frac{x^2}{h^2}+\frac{y^2}{k^2}, \qquad x=0, \quad y=0. \tag{1}$$

чертёж при $a=b=h=k=1$:



$x\geqslant 0$, $y\geqslant 0$

координаты:

$$\begin{cases} x=ar\cos^{2/3}(\varphi)\,, \\ y=br\sin^{2/3}(\varphi)\,. \end{cases} \tag{2}$$

Якобиан:

$$J=\begin{vmatrix} x'_r & x'_\varphi \\ y'_r & y'_\varphi \end{vmatrix}=\begin{vmatrix} a\cos^{2/3}(\varphi) & -\frac{2}{3}ar\cos^{-1/3}(\varphi)\sin(\varphi) \\ b\sin^{2/3}(\varphi) & \frac{2}{3}br\sin^{-1/3}(\varphi)\cos(\varphi) \end{vmatrix}=\frac{2}{3}abr\begin{vmatrix} \cos^{2/3}(\varphi) & -\cos^{-1/3}(\varphi)\sin(\varphi) \\ \sin^{2/3}(\varphi) & \sin^{-1/3}(\varphi)\cos(\varphi) \end{vmatrix}=$$

$$= \frac{2}{3}abr\left\{\cos^{2/3}(\varphi)\sin^{-1/3}(\varphi)\cos(\varphi)+\sin^{2/3}(\varphi)\cos^{-1/3}(\varphi)\sin(\varphi)\right\} = \frac{2}{3}abr\left\{\cos^{5/3}(\varphi)\sin^{-1/3}(\varphi)+\sin^{5/3}(\varphi)\cos^{-1/3}(\varphi)\right\} =$$
$$= \frac{2}{3}abr\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi)\left\{\cos^{6/3}(\varphi)+\sin^{6/3}(\varphi)\right\} = \frac{2}{3}abr\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi). \tag{3}$$

Уравнение кривой (R - значение r на границе фигуры):

$$\frac{\left(aR\cos^{2/3}(\varphi)\right)^3}{a^3}+\frac{\left(bR\sin^{2/3}(\varphi)\right)^3}{b^3} = \frac{\left(aR\cos^{2/3}(\varphi)\right)^2}{h^2}+\frac{\left(bR\sin^{2/3}(\varphi)\right)^2}{k^2}, \tag{4}$$

$$R^3\cos^2(\varphi)+R^3\sin^2(\varphi) = \frac{a^2R^2\cos^{4/3}(\varphi)}{h^2}+\frac{b^2R^2\sin^{4/3}(\varphi)}{k^2}, \tag{5}$$

$$R^3 = R^2\left(\frac{a^2}{h^2}\cos^{4/3}(\varphi)+\frac{b^2}{k^2}\sin^{4/3}(\varphi)\right), \tag{6}$$

$$R = \frac{a^2}{h^2}\cos^{4/3}(\varphi)+\frac{b^2}{k^2}\sin^{4/3}(\varphi). \tag{7}$$

Интеграл площади:

$$S = \iint\limits_{\Omega} 1\cdot dxdy = \int\limits_0^{\pi/2} d\varphi\int\limits_0^R |J|\,dr, \tag{8}$$

по r:

$$\int\limits_0^R |J|\,dr = \int\limits_0^R \frac{2}{3}abr\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi)\,dr = \frac{2}{3}ab\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi)\int\limits_0^R rdr =$$
$$= \frac{2}{3}ab\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi)\left.\frac{r^2}{2}\right|_0^R = \frac{1}{3}abR^2\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi) =$$
$$= \frac{1}{3}ab\left(\frac{a^2}{h^2}\cos^{4/3}(\varphi)+\frac{b^2}{k^2}\sin^{4/3}(\varphi)\right)^2\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi) = \tag{9}$$
$$= \frac{1}{3}ab\left(\frac{a^4}{h^4}\cos^{8/3}(\varphi)+2\frac{a^2}{h^2}\frac{b^2}{k^2}\cos^{4/3}(\varphi)\sin^{4/3}(\varphi)+\frac{b^4}{k^4}\sin^{8/3}(\varphi)\right)\sin^{-1/3}(\varphi)\cos^{-1/3}(\varphi) =$$
$$= \frac{1}{3}ab\left(\frac{a^4}{h^4}\cos^{7/3}(\varphi)\sin^{-1/3}(\varphi)+2\frac{a^2}{h^2}\frac{b^2}{k^2}\cos(\varphi)\sin(\varphi)+\frac{b^4}{k^4}\sin^{7/3}(\varphi)\cos^{-1/3}(\varphi)\right).$$

Теперь по φ:

$$S = \int\limits_0^{\pi/2} d\varphi\int\limits_0^R |J|\,dr =$$
$$= \frac{ab}{3}\int\limits_0^{\pi/2}\left(\frac{a^4}{h^4}\cos^{7/3}(\varphi)\sin^{-1/3}(\varphi)+2\frac{a^2}{h^2}\frac{b^2}{k^2}\cos(\varphi)\sin(\varphi)+\frac{b^4}{k^4}\sin^{7/3}(\varphi)\cos^{-1/3}(\varphi)\right)d\varphi = \tag{10}$$

$$= \frac{ab}{3}\left(\frac{a^4}{h^4}\int\limits_0^{\pi/2}\cos^{7/3}(\varphi)\sin^{-1/3}(\varphi)\,d\varphi + 2\frac{a^2}{h^2}\frac{b^2}{k^2}\int\limits_0^{\pi/2}\cos(\varphi)\sin(\varphi)\,d\varphi + \frac{b^4}{k^4}\int\limits_0^{\pi/2}\sin^{7/3}(\varphi)\cos^{-1/3}(\varphi)\,d\varphi\right).$$

Первое слагаемое (10):

$$\int\limits_0^{\pi/2}\cos^{7/3}\varphi\sin^{-1/3}\varphi d\varphi = \int\limits_0^{\pi/2}\cos^{4/3}\varphi\sin^{-4/3}\varphi\cos\varphi\sin\varphi d\varphi = \int\limits_0^{\pi/2}\left(\cos^2\varphi\right)^{2/3}\left(\sin^2\varphi\right)^{-2/3}\cos\varphi\sin\varphi d\varphi =$$

$\sin^2\varphi = t$

$$= \frac{1}{2}\int\limits_0^{\pi/2}\left(1-\sin^2\varphi\right)^{2/3}\left(\sin^2\varphi\right)^{-2/3}\left(\sin^2\varphi\right)'d\varphi = \frac{1}{2}\int\limits_0^{1}(1-t)^{2/3}t^{-2/3}dt \tag{11}$$

Подстановка Чебышова: $p = -\frac{2}{3}$, $q = 1$, $r = \frac{2}{3}$

$$\frac{p+1}{q} = \frac{-\frac{2}{3}+1}{1} = -\frac{2}{3}+1 = \frac{1}{3} \notin \mathbb{Z} \tag{12}$$

$$\frac{p+1}{q} + r = -\frac{2}{3}+1+\frac{2}{3} = 1 \in \mathbb{Z} \tag{13}$$

Третий случай: $\frac{1}{t} - 1 = z^3$, $t = \frac{1}{z^3+1}$

$$\int\limits_0^{1}(1-t)^{2/3}t^{-2/3}dt = \int\limits_0^{1}\left(\frac{1}{t}-1\right)^{2/3}dt = \int\limits_\infty^{0}\left(z^3\right)^{2/3}d\frac{1}{z^3+1} = -\int\limits_0^{\infty}z^2 d\frac{1}{z^3+1} =$$

$$= -\left.\frac{z^2}{z^3+1}\right|_0^\infty + \int\limits_0^\infty\frac{1}{z^3+1}dz^2 = 2\int\limits_0^\infty\frac{z}{z^3+1}dz. \tag{14}$$

Из (11) и (14):

$$\int\limits_0^{\pi/2}\cos^{7/3}\varphi\sin^{-1/3}\varphi d\varphi = \frac{1}{2}\int\limits_0^{1}(1-t)^{2/3}t^{-2/3}dt = \int\limits_0^\infty\frac{z}{z^3+1}dz. \tag{15}$$

$$\frac{z}{z^3+1} = \frac{z}{(z+1)(z^2-z+1)} = \frac{A}{z+1} + \frac{Bz+C}{z^2-z+1} = \frac{A\left(z^2-z+1\right)+(Bz+C)(z+1)}{(z+1)(z^2-z+1)} \tag{16}$$

$$z = A\left(z^2-z+1\right)+(Bz+C)(z+1) = Az^2 - Az + A + Bz^2 + Bz + Cz + C \tag{17}$$

$$\begin{cases} A+B=0 \\ -A+B+C=1 \\ A+C=0 \end{cases} \tag{18}$$

$$B = C = -A, \quad -A-A-A=1 \tag{19}$$

$$A = -\frac{1}{3},\ B = C = \frac{1}{3} \tag{20}$$

$$\frac{z}{z^3+1} = -\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{z+1}{z^2-z+1} \tag{21}$$

$$\int \frac{z}{z^3+1}dz = \int \left\{-\frac{1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{z+1}{z^2-z+1}\right\}dz = -\frac{1}{3}\ln|z+1| + \frac{1}{3}\int \frac{z+1}{z^2-z+1}dz, \tag{22}$$

$$z^2-z+1 = \left(z^2-2z\frac{1}{2}+\frac{1}{4}\right) - \frac{1}{4} + 1 = \left(z-\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{2z-1}{\sqrt{3}}\right)^2+1\right]. \tag{23}$$

$\frac{2z-1}{\sqrt{3}} = t$, $z = \frac{\sqrt{3}t+1}{2}$, $dz = \frac{\sqrt{3}}{2}dt$

$$\int \frac{z+1}{z^2-z+1}dz = \int \frac{z+1}{\frac{3}{4}\left[\left(\frac{2z-1}{\sqrt{3}}\right)^2+1\right]}dz = \frac{4}{3}\int \frac{\frac{\sqrt{3}t+1}{2}+1}{t^2+1}\frac{\sqrt{3}}{2}dt = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}t+3}{t^2+1}dt =$$

$$= \frac{1}{\sqrt{3}}\left(\sqrt{3}\int \frac{t}{t^2+1}dt + 3\int\frac{dt}{t^2+1}\right) = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\ln\left(t^2+1\right) + 3\operatorname{arctg} t\right) + C = \frac{1}{2}\ln\left(t^2+1\right) + \sqrt{3}\operatorname{arctg} t + C = \tag{24}$$

$$= \frac{1}{2}\ln\left[\left(\frac{2z-1}{\sqrt{3}}\right)^2+1\right] + \sqrt{3}\operatorname{arctg}\frac{2z-1}{\sqrt{3}} + C,$$

$$\int \frac{z}{z^3+1}dz = -\frac{1}{3}\ln|z+1| + \frac{1}{3}\int\frac{z+1}{z^2-z+1}dz = -\frac{1}{3}\ln|z+1| + \frac{1}{3}\left\{\frac{1}{2}\ln\left[\left(\frac{2z-1}{\sqrt{3}}\right)^2+1\right] + \sqrt{3}\operatorname{arctg}\frac{2z-1}{\sqrt{3}} + C\right\} =$$

$$= \frac{1}{6}\left\{\ln\left[\left(\frac{2z-1}{\sqrt{3}}\right)^2+1\right] - \ln|z+1|^2\right\} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2z-1}{\sqrt{3}} + \frac{C}{3} = \tag{25}$$

$$= \frac{1}{6}\ln\left[\frac{\left(\frac{2z-1}{\sqrt{3}}\right)^2+1}{|z+1|^2}\right] + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2z-1}{\sqrt{3}} + \frac{C}{3},$$

$$\int\limits_0^\infty \frac{z}{z^3+1}dz = \frac{1}{6}\ln\left[\frac{\left(\frac{2z-1}{\sqrt{3}}\right)^2+1}{|z+1|^2}\right] + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2z-1}{\sqrt{3}}\Bigg|_0^\infty = \frac{1}{6}\left(\ln\frac{4}{3} - \ln\frac{4}{3}\right) + \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \operatorname{arctg}\left(-\frac{1}{\sqrt{3}}\right)\right] =$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2}+\frac{\pi}{6}\right] = \frac{1}{\sqrt{3}}\frac{4\pi}{6} = \frac{2\pi}{3\sqrt{3}}. \tag{26}$$

Из (15)

$$\int\limits_0^{\pi/2} \cos^{7/3}\varphi\sin^{-1/3}\varphi d\varphi = \int\limits_0^\infty \frac{z}{z^3+1}dz = \frac{2\pi}{3\sqrt{3}} \tag{27}$$

Третье слагаемое (10) ($\varphi = \frac{\pi}{2} - \psi$):

$$\int\limits_0^{\pi/2} \sin^{7/3}\varphi\cos^{-1/3}\varphi d\varphi = -\int\limits_{\pi/2}^0 \sin^{7/3}\left(\frac{\pi}{2}-\psi\right)\cos^{-1/3}\left(\frac{\pi}{2}-\psi\right)d\psi = \int\limits_0^{\pi/2}\cos^{7/3}\psi\sin^{-1/3}\psi d\psi = \frac{2\pi}{3\sqrt{3}} \tag{28}$$

Второе слагаемое (10)

$$\int\limits_0^{\pi/2} \cos(\varphi)\sin(\varphi)\,d\varphi = \frac{1}{2}\int\limits_0^{\pi/2} \sin(2\varphi)\,d\varphi = -\frac{1}{4}\cos(2\varphi)\Big|_0^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}. \tag{29}$$

Тогда, окончательно, сама площадь равна

$$S = \frac{ab}{3}\left(\frac{a^4}{h^4}\frac{2\pi}{3\sqrt{3}} + 2\frac{a^2}{h^2}\frac{b^2}{k^2}\frac{1}{2} + \frac{b^4}{k^4}\frac{2\pi}{3\sqrt{3}}\right) = \frac{ab}{3}\left[\frac{2\pi}{3\sqrt{3}}\left(\frac{a^4}{h^4} + \frac{b^4}{k^4}\right) + \frac{a^2}{h^2}\frac{b^2}{k^2}\right]. \tag{30}$$