1 Теория

$$J\left[y\right]=\int\limits_{x_1}^{x_2}F\left(x,y,y'\right)dx, \tag{1}$$

экстремали с одной угловой точкой: $A\left(x_1,y_1\right)\longrightarrow M\left(x_0,y_0\right)\longrightarrow B\left(x_2,y_2\right)$.

$$F'_y-\frac{d}{dx}F'_{y'}=0 \tag{2}$$

$$\delta J\left[y\right]=\delta\int\limits_{x_1}^{x_0}F\left(x,y,y'\right)dx+\delta\int\limits_{x_0}^{x_2}F\left(x,y,y'\right)dx, \tag{3}$$

по формуле, выведенной в прошлый раз:

$$\delta J\left[y\right]=\left.\left(F-F'_{y'}y'\right)\right|_{x=x_2}\delta x_2-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_1}\delta x_1+\left.F'_{y'}\right|_{x=x_2}\delta y_2-\left.F'_{y'}\right|_{x=x_1}\delta y_1. \tag{4}$$

$$\begin{aligned}\delta J\left[y\right]&=\delta\int\limits_{x_1}^{x_0}F\left(x,y,y'\right)dx+\delta\int\limits_{x_0}^{x_2}F\left(x,y,y'\right)dx= \\ &=\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0-0}\delta x_0+\left.F'_{y'}\right|_{x=x_0-0}\delta y_0+ \\ &-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0+0}\delta x_0-\left.F'_{y'}\right|_{x=x_0+0}\delta y_0= \\ &=\left[\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0-0}-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0+0}\right]\delta x_0+\left[\left.F'_{y'}\right|_{x=x_0-0}-\left.F'_{y'}\right|_{x=x_0+0}\right]\delta y_0\end{aligned} \tag{5}$$

На экстремали $\delta J\left[y\right]=0$,

$$\left[\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0-0}-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0+0}\right]\delta x_0+\left[\left.F'_{y'}\right|_{x=x_0-0}-\left.F'_{y'}\right|_{x=x_0+0}\right]\delta y_0=0 \tag{6}$$

1) Точка $M\left(x_0,y_0\right)$ движется свободно:

$$\left\{\begin{array}{l}\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0-0}-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0+0}=0, \\ \left.F'_{y'}\right|_{x=x_0-0}-\left.F'_{y'}\right|_{x=x_0+0}=0,\end{array}\right. \tag{7}$$

условия Вейерштрасса-Эрдмана.

2) $y_0=\varphi\left(x_0\right)$: $\delta y_0=\varphi'\delta x_0$,

$$\left[\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0-0}-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0+0}\right]\delta x_0+\left[\left.F'_{y'}\right|_{x=x_0-0}-\left.F'_{y'}\right|_{x=x_0+0}\right]\varphi'\delta x_0=0, \tag{8}$$

$$\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0-0}-\left.\left(F-F'_{y'}y'\right)\right|_{x=x_0+0}+\left[\left.F'_{y'}\right|_{x=x_0-0}-\left.F'_{y'}\right|_{x=x_0+0}\right]\varphi'=0, \tag{9}$$

$$\left.\left(F-F'_{y'}y'+\varphi'F'_{y'}\right)\right|_{x=x_0-0}-\left.\left(F-F'_{y'}y'+\varphi'F'_{y'}\right)\right|_{x=x_0+0}=0, \tag{10}$$

$$\left.\left[F+\left(\varphi'-y'\right)F'_{y'}\right]\right|_{x=x_0-0}=\left.\left[F+\left(\varphi'-y'\right)F'_{y'}\right]\right|_{x=x_0+0}. \tag{11}$$

2 Пример: Задача 6.1

$$J\left[y\right] = \int\limits_0^4 \left(y' - 1\right)^2 \left(y' + 1\right)^2 dx, \qquad y\left(0\right) = 0, \quad y\left(4\right) = 2. \tag{12}$$

Следствия уравнения Эйлера

$$F = \left(y' - 1\right)^2 \left(y' + 1\right)^2 = F\left(y'\right) \tag{13}$$

$$y' = A, \qquad y = Ax + B \tag{14}$$

$$y = \left\{ \begin{array}{ll} C_1 x + C_2, & x < x_0 \\ C_3 x + C_4, & x > x_0 \end{array} \right. \tag{15}$$

Учёт границ

$$y\left(0\right) = C_2 = 0, \quad y\left(4\right) = 4C_3 + C_4 = 2 \tag{16}$$

$$C_2 = 0, \quad C_4 = 2 - 4C_3 \tag{17}$$

$$y = \left\{ \begin{array}{ll} C_1 x, & x < x_0 \\ C_3 x + 2 - 4C_3, & x > x_0 \end{array} \right. \tag{18}$$

Учёт непрерывности

$$C_1 x_0 = C_3 x_0 + 2 - 4C_3 \tag{19}$$

Условия Вейерштрасса-Эрдмана

$$\left(F - y' F'_{y'}\right)\Big|_{x = x_0 - 0} - \left(F - y' F'_{y'}\right)\Big|_{x = x_0 + 0} = 0 \tag{20}$$

$$F'_{y'}\Big|_{x = x_0 - 0} - F'_{y'}\Big|_{x = x_0 + 0} = 0 \tag{21}$$

$$F = \left(y' - 1\right)^2 \left(y' + 1\right)^2 = \left(y'^2 - 1\right)^2 = y'^4 - 2y'^2 + 1 \tag{22}$$

$$F'_{y'} = 4y'^3 - 4y' \tag{23}$$

$$F - y' F'_{y'} = \left(y'^4 - 2y'^2 + 1\right) - y' \left(4y'^3 - 4y'\right) = y'^4 - 2y'^2 + 1 - 4y'^4 + 4y'^2 = 1 - 3y'^4 + 2y'^2 \tag{24}$$

$$\left\{ \begin{array}{l} \left(1 - 3y'^4 + 2y'^2\right)\Big|_{x = x_0 - 0} - \left(1 - 3y'^4 + 2y'^2\right)\Big|_{x = x_0 + 0} = 0 \\ \left(4y'^3 - 4y'\right)\Big|_{x = x_0 - 0} - \left(4y'^3 - 4y'\right)\Big|_{x = x_0 + 0} = 0 \end{array} \right. \tag{25}$$

$$\left\{ \begin{array}{l} \left(1 - 3C_1^4 + 2C_1^2\right) - \left(1 - 3C_3^4 + 2C_3^2\right) = 0 \\ \left(4C_1^3 - 4C_1\right) - \left(4C_3^3 - 4C_3\right) = 0 \end{array} \right. \tag{26}$$

$$\left\{ \begin{array}{l} 2C_1^2 - 2C_3^2 + 3C_3^4 - 3C_1^4 = 0 \\ C_1^3 - C_3^3 - C_1 + C_3 = 0 \end{array} \right. \tag{27}$$

$$\left\{ \begin{array}{l} 2\left(C_1^2 - C_3^2\right) - 3\left(C_1^4 - C_3^4\right) = 0 \\ \left(C_1^3 - C_3^3\right) - \left(C_1 - C_3\right) = 0 \end{array} \right. \tag{28}$$

$$\left\{ \begin{array}{l} \left(C_1 - C_3\right)\left(C_1 + C_3\right)\left[2 - 3\left(C_1^2 + C_3^2\right)\right] = 0 \\ \left(C_1 - C_3\right)\left[C_1^2 + C_1 C_3 + C_3^2 - 1\right] = 0 \end{array} \right. \tag{29}$$

Или $C_1 = C_3$,

$$C_1 x_0 = C_1 x_0 + 2 - 4C_1 \quad \Longrightarrow \quad C_1 = \frac{1}{2} \tag{30}$$

$$y = \begin{cases} \frac{1}{2}x, & x < x_0 \\ \frac{1}{2}x + 2 - 4 \cdot \frac{1}{2}, & x > x_0 \end{cases} = \frac{1}{2}x \tag{31}$$

Решение без угловой точки.

Или $C_3 = -C_1$,

$$C_1^2 - 1 = 0, \tag{32}$$

$$C_1 x_0 = -C_1 x_0 + 2 + 4C_1, \tag{33}$$

$$2C_1 x_0 = 2 + 4C_1 \quad \Longrightarrow \quad x_0 = \frac{1}{C_1} + 2, \tag{34}$$

$$y = \begin{cases} C_1 x, & x < x_0 \\ -C_1 x + 2 + 4C_1. & x > x_0 \end{cases} \tag{35}$$

$C_1 = 1$, $C_3 = -1$:

$$x_0 = 1 + 2 = 3, \tag{36}$$

$$y = \begin{cases} x, & x < 3, \\ -x + 6, & x > 3. \end{cases} \tag{37}$$

$C_1 = -1$, $C_3 = 1$:

$$x_0 = -1 + 2 = 1, \tag{38}$$

$$y = \begin{cases} -x, & x < 1, \\ x - 2, & x > 1. \end{cases} \tag{39}$$