$$\vec{e}_p \equiv \frac{\vec{r}'_p}{H_p} = \frac{1}{H_p}\begin{pmatrix} x'_p \\ y'_p \\ z'_p \end{pmatrix}, \quad \vec{e}_q \equiv \ldots, \tag{1}$$

$$[\vec{e}_p \times \vec{e}_q] = \vec{e}_s, \quad [\vec{e}_q \times \vec{e}_s] = \vec{e}_p, \quad [\vec{e}_s \times \vec{e}_p] = \vec{e}_q \tag{2}$$

1 Дифференциальные свойства ортов

$$\vec{e}_\beta \cdot \frac{\partial}{\partial\gamma}\vec{e}_\alpha = -\vec{e}_\alpha \cdot \frac{\partial}{\partial\gamma}\vec{e}_\beta \tag{3}$$

$$\vec{e}_\alpha \cdot \frac{\partial}{\partial\gamma}\vec{e}_\alpha = 0. \tag{4}$$

$$\vec{e}_\beta \cdot \frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{1}{H_\alpha}\frac{\partial}{\partial\alpha}H_\beta, \quad \alpha \neq \beta \tag{5}$$

$$\vec{e}_\gamma \cdot \frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{H_\beta}{H_\alpha}\vec{e}_\gamma \cdot \frac{\partial}{\partial\alpha}\vec{e}_\beta, \quad \gamma \neq \alpha,\ \gamma \neq \beta. \tag{6}$$

2 Ротор

Ротор считаем покомпонентно

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \vec{i} & \partial/\partial x & P \\ \vec{j} & \partial/\partial y & Q \\ \vec{k} & \partial/\partial z & R \end{vmatrix} = \vec{i}\left(R'_y - Q'_z\right) + \vec{j}\left(P'_z - R'_x\right) + \vec{k}\left(Q'_x - P'_y\right). \tag{7}$$

$$\begin{aligned}
\vec{e}_p \cdot \operatorname{rot}\vec{F} &= \frac{x'_p}{H_p}\left(R'_y - Q'_z\right) + \frac{y'_p}{H_p}\left(P'_z - R'_x\right) + \frac{z'_p}{H_p}\left(Q'_x - P'_y\right) = \\
&= \frac{1}{H_p}\left[x'_pR'_y - x'_pQ'_z + y'_pP'_z - y'_pR'_x + z'_pQ'_x - z'_pP'_y\right] = \\
&= \frac{1}{H_p}\left[\left(x'_pR'_y - y'_pR'_x\right) + \left(y'_pP'_z - z'_pP'_y\right) + \left(z'_pQ'_x - x'_pQ'_z\right)\right] = \\
&= \frac{1}{H_p}\left[\left(x'_pp'_y - y'_pp'_x\right)R'_p + \left(x'_pq'_y - y'_pq'_x\right)R'_q + \left(x'_ps'_y - y'_ps'_x\right)R'_s + \right. \\
&+ \left(y'_pp'_z - z'_pp'_y\right)P'_p + \left(y'_pq'_z - z'_pq'_y\right)P'_q + \left(y'_ps'_z - z'_ps'_y\right)P'_s + \\
&+ \left.\left(z'_pp'_x - x'_pp'_z\right)Q'_p + \left(z'_pq'_x - x'_pq'_z\right)Q'_q + \left(z'_ps'_x - x'_ps'_z\right)Q'_s\right] = \\
&= \frac{1}{H_p}\left[\begin{vmatrix} x'_p & p'_x & P'_p \\ y'_p & p'_y & Q'_p \\ z'_p & p'_z & R'_p \end{vmatrix} + \begin{vmatrix} x'_p & q'_x & P'_q \\ y'_p & q'_y & Q'_q \\ z'_p & q'_z & R'_q \end{vmatrix} + \begin{vmatrix} x'_p & s'_x & P'_s \\ y'_p & s'_y & Q'_s \\ z'_p & s'_z & R'_s \end{vmatrix}\right] = \\
&= \frac{1}{H_p}\left[\left(\vec{r}'_p, \operatorname{grad} p, \vec{F}'_p\right) + \left(\vec{r}'_p, \operatorname{grad} q, \vec{F}'_q\right) + \left(\vec{r}'_p, \operatorname{grad} s, \vec{F}'_s\right)\right] = \\
&= \left(\vec{e}_p, \frac{\vec{e}_p}{H_p}, \vec{F}'_p\right) + \left(\vec{e}_p, \frac{\vec{e}_q}{H_q}, \vec{F}'_q\right) + \left(\vec{e}_p, \frac{\vec{e}_s}{H_s}, \vec{F}'_s\right) = \\
&= \frac{1}{H_p}\vec{F}'_p \cdot [\vec{e}_p \times \vec{e}_p] + \frac{1}{H_q}\vec{F}'_q \cdot [\vec{e}_p \times \vec{e}_q] + \frac{1}{H_s}\vec{F}'_s \cdot [\vec{e}_p \times \vec{e}_s] = \\
&= \frac{1}{H_q}\vec{F}'_q \cdot \vec{e}_s - \frac{1}{H_s}\vec{F}'_s \cdot \vec{e}_q;
\end{aligned} \tag{8}$$

$$\vec{F}'_q\vec{e}_s = \frac{\partial}{\partial q}\left(F_p\vec{e}_p + F_q\vec{e}_q + F_s\vec{e}_s\right)\vec{e}_s =$$

$$= \left(\frac{\partial}{\partial q}F_p\vec{e}_p + \frac{\partial}{\partial q}F_q\vec{e}_q + \frac{\partial}{\partial q}F_s\vec{e}_s + F_p\frac{\partial}{\partial q}\vec{e}_p + F_q\frac{\partial}{\partial q}\vec{e}_q + F_s\frac{\partial}{\partial q}\vec{e}_s\right)\vec{e}_s =$$

по (3)

$$= \frac{\partial}{\partial q}F_s + F_p\vec{e}_s \cdot \frac{\partial}{\partial q}\vec{e}_p + F_q\vec{e}_s \cdot \frac{\partial}{\partial q}\vec{e}_q = \frac{\partial}{\partial q}F_s + F_p\vec{e}_s \cdot \frac{\partial}{\partial q}\vec{e}_p - F_q\vec{e}_q\frac{\partial}{\partial q}\vec{e}_s = \tag{9}$$

по (5)

$$= \frac{\partial}{\partial q}F_s + F_p \vec{e}_s \frac{\partial}{\partial q}\vec{e}_p - F_q \frac{1}{H_s}\frac{\partial}{\partial s}H_q,$$

и аналогично,

$$\vec{F}_s' \vec{e}_q = \frac{\partial}{\partial s}F_q + F_p \vec{e}_q \frac{\partial}{\partial s}\vec{e}_p - F_s \frac{1}{H_q}\frac{\partial}{\partial q}H_s. \tag{10}$$

Возвращаемся:

$$\vec{e}_p \cdot \operatorname{rot}\vec{F} = \frac{1}{H_q}\vec{F}_q' \vec{e}_s - \frac{1}{H_s}\vec{F}_s' \vec{e}_q = \frac{1}{H_q}\left(\frac{\partial}{\partial q}F_s + F_p \vec{e}_s \frac{\partial}{\partial q}\vec{e}_p - F_q \frac{1}{H_s}\frac{\partial}{\partial s}H_q\right) - \frac{1}{H_s}\left(\frac{\partial}{\partial s}F_q + F_p \vec{e}_q \frac{\partial}{\partial s}\vec{e}_p - F_s \frac{1}{H_q}\frac{\partial}{\partial q}H_s\right) =$$

без е и с е:

$$= \left(\frac{1}{H_q}\frac{\partial}{\partial q}F_s - \frac{1}{H_s}\frac{\partial}{\partial s}F_q + F_s \frac{1}{H_q H_s}\frac{\partial}{\partial q}H_s - F_q \frac{1}{H_q H_s}\frac{\partial}{\partial s}H_q\right) + \left(\frac{1}{H_q}F_p \vec{e}_s \cdot \frac{\partial}{\partial q}\vec{e}_p - \frac{1}{H_s}F_p \vec{e}_q \cdot \frac{\partial}{\partial s}\vec{e}_p\right) =$$

$$= \frac{1}{H_q H_s}\left(H_s \frac{\partial}{\partial q}F_s + F_s \frac{\partial}{\partial q}H_s - H_q \frac{\partial}{\partial s}F_q - F_q \frac{\partial}{\partial s}H_q\right) + \frac{1}{H_q H_s}F_p \left(H_s \vec{e}_s \cdot \frac{\partial}{\partial q}\vec{e}_p - H_q \vec{e}_q \cdot \frac{\partial}{\partial s}\vec{e}_p\right) = \tag{11}$$

собираем и переставляем:

$$= \frac{1}{H_q H_s}\left(\frac{\partial}{\partial q}(H_s F_s) - \frac{\partial}{\partial s}(H_q F_q)\right) - \frac{1}{H_q H_s}F_p \left(H_s \vec{e}_p \cdot \frac{\partial}{\partial q}\vec{e}_s - H_q \vec{e}_p \cdot \frac{\partial}{\partial s}\vec{e}_q\right).$$

Из (6) при $\gamma = p$, $\beta = s$, $\alpha = q$:

$$\vec{e}_p \cdot \frac{\partial}{\partial s}\vec{e}_q = \frac{H_s}{H_q}\vec{e}_p \cdot \frac{\partial}{\partial q}\vec{e}_s. \tag{12}$$

Тогда

$$\vec{e}_p \cdot \operatorname{rot}\vec{F} = \frac{1}{H_q H_s}\left(\frac{\partial}{\partial q}(H_s F_s) - \frac{\partial}{\partial s}(H_q F_q)\right) - \frac{1}{H_q H_s}F_p \vec{e}_p \cdot \left(H_s \frac{\partial}{\partial q}\vec{e}_s - H_q \frac{H_s}{H_q}\frac{\partial}{\partial q}\vec{e}_s\right) =$$

$$= \frac{1}{H_q H_s}\left(\frac{\partial}{\partial q}(H_s F_s) - \frac{\partial}{\partial s}(H_q F_q)\right). \tag{13}$$

Аналогично,

$$\vec{e}_q \cdot \operatorname{rot}\vec{F} = \frac{1}{H_p H_s}\left(\frac{\partial}{\partial s}(H_p F_p) - \frac{\partial}{\partial p}(H_s F_s)\right), \tag{14}$$

$$\vec{e}_s \cdot \operatorname{rot}\vec{F} = \frac{1}{H_p H_q}\left(\frac{\partial}{\partial p}(H_q F_q) - \frac{\partial}{\partial q}(H_p F_p)\right), \tag{15}$$

и тогда

$$\operatorname{rot}\vec{F} = \vec{e}_p\left(\vec{e}_p \cdot \operatorname{rot}\vec{F}\right) + \vec{e}_q\left(\vec{e}_q \cdot \operatorname{rot}\vec{F}\right) + \vec{e}_s\left(\vec{e}_s \cdot \operatorname{rot}\vec{F}\right) =$$

$$= \vec{e}_p \frac{1}{H_q H_s}\left(\frac{\partial}{\partial q}(H_s F_s) - \frac{\partial}{\partial s}(H_q F_q)\right) + \vec{e}_q \frac{1}{H_p H_s}\left(\frac{\partial}{\partial s}(H_p F_p) - \frac{\partial}{\partial p}(H_s F_s)\right) + \vec{e}_s \frac{1}{H_p H_q}\left(\frac{\partial}{\partial p}(H_q F_q) - \frac{\partial}{\partial q}(H_p F_p)\right) =$$

$$= \frac{1}{H_p H_q H_s}\left[\vec{e}_p H_p \left(\frac{\partial}{\partial q}(H_s F_s) - \frac{\partial}{\partial s}(H_q F_q)\right) - \vec{e}_q H_q \left(\frac{\partial}{\partial p}(H_s F_s) - \frac{\partial}{\partial s}(H_p F_p)\right) + \vec{e}_s H_s \left(\frac{\partial}{\partial p}(H_q F_q) - \frac{\partial}{\partial q}(H_p F_p)\right)\right] = \tag{16}$$

$$= \frac{1}{H_p H_q H_s}\begin{vmatrix} H_p \vec{e}_p & \partial/\partial p & H_p F_p \\ H_q \vec{e}_q & \partial/\partial q & H_q F_q \\ H_s \vec{e}_s & \partial/\partial s & H_s F_s \end{vmatrix}.$$

Пример №193 Найти ротор в цилиндрических координатах:

$$\vec{F} = \cos\varphi \vec{e}_r - \frac{\sin\varphi}{r}\vec{e}_\varphi + r^2 \vec{e}_z. \tag{17}$$

В цилиндрических координатах

$$H_r = 1, \quad H_\varphi = r, \quad H_z = 1,$$

$$\operatorname{rot}\vec{F} = \frac{1}{H_r H_\varphi H_z}\begin{vmatrix} H_r \vec{e}_r & \partial/\partial r & H_r F_r \\ H_\varphi \vec{e}_\varphi & \partial/\partial \varphi & H_\varphi F_\varphi \\ H_z \vec{e}_z & \partial/\partial z & H_z F_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & \partial/\partial r & F_r \\ r\vec{e}_\varphi & \partial/\partial \varphi & rF_\varphi \\ \vec{e}_z & \partial/\partial z & F_z \end{vmatrix}. \tag{18}$$

В нашем случае

$$\operatorname{rot}\vec{F}=\frac{1}{r}\begin{vmatrix}\vec{e}_r & \partial/\partial r & \cos\varphi\\ r\vec{e}_\varphi & \partial/\partial\varphi & -\frac{\sin\varphi}{r}r\\ \vec{e}_z & \partial/\partial z & r^2\end{vmatrix}=\frac{1}{r}\begin{vmatrix}\vec{e}_r & \partial/\partial r & \cos\varphi\\ r\vec{e}_\varphi & \partial/\partial\varphi & -\sin\varphi\\ \vec{e}_z & \partial/\partial z & r^2\end{vmatrix}=$$

$$=\frac{1}{r}\left\{\vec{e}_r\left[\frac{\partial}{\partial\varphi}\left(r^2\right)-\frac{\partial}{\partial z}\left(-\sin\varphi\right)\right]-r\vec{e}_\varphi\left[\frac{\partial}{\partial r}\left(r^2\right)-\frac{\partial}{\partial z}\cos\varphi\right]+\vec{e}_z\left[\frac{\partial}{\partial r}\left(-\sin\varphi\right)-\frac{\partial}{\partial\varphi}\cos\varphi\right]\right\}= \tag{19}$$

$$=\frac{1}{r}\left\{-r\vec{e}_\varphi\left[\frac{\partial}{\partial r}\left(r^2\right)\right]+\vec{e}_z\left[-\frac{\partial}{\partial\varphi}\cos\varphi\right]\right\}=\frac{1}{r}\left\{-r\vec{e}_\varphi\left[2r\right]+\vec{e}_z\left[\sin\varphi\right]\right\}=-2r\vec{e}_\varphi+\frac{\sin\varphi}{r}\vec{e}_z$$

3 Лапласиан

$$\operatorname{grad}u=\frac{u'_p}{H_p}\vec{e}_p+\frac{u'_q}{H_q}\vec{e}_q+\frac{u'_s}{H_s}\vec{e}_s, \tag{20}$$

$$\operatorname{div}\vec{F}=\frac{1}{H_pH_qH_s}\left[\frac{\partial}{\partial p}\left(H_qH_sF_p\right)+\frac{\partial}{\partial q}\left(H_pH_sF_q\right)+\frac{\partial}{\partial s}\left(H_pH_qF_s\right)\right]; \tag{21}$$

Подставляем одно в другое:

$$\Delta u=\operatorname{div}\left(\operatorname{grad}u\right)=\operatorname{div}\left(\frac{u'_p}{H_p}\vec{e}_p+\frac{u'_q}{H_q}\vec{e}_q+\frac{u'_s}{H_s}\vec{e}_s\right)=\frac{1}{H_pH_qH_s}\left[\frac{\partial}{\partial p}\left(H_qH_s\frac{u'_p}{H_p}\right)+\frac{\partial}{\partial q}\left(H_pH_s\frac{u'_q}{H_q}\right)+\frac{\partial}{\partial s}\left(H_pH_q\frac{u'_s}{H_s}\right)\right]=$$

$$=\frac{1}{H_pH_qH_s}\left[\frac{H_qH_s}{H_p}\frac{\partial}{\partial p}\left(u'_p\right)+\frac{H_pH_s}{H_q}\frac{\partial}{\partial q}\left(u'_q\right)+\frac{H_pH_q}{H_s}\frac{\partial}{\partial s}\left(u'_s\right)+u'_p\frac{\partial}{\partial p}\left(\frac{H_qH_s}{H_p}\right)+u'_q\frac{\partial}{\partial q}\left(\frac{H_pH_s}{H_q}\right)+u'_s\frac{\partial}{\partial s}\left(\frac{H_pH_q}{H_s}\right)\right]= \tag{22}$$

$$=\left[\frac{1}{H_p^2}u''_{pp}+\frac{1}{H_q^2}u''_{qq}+\frac{1}{H_s^2}u''_{ss}\right]+\frac{1}{H_pH_qH_s}\left[u'_p\frac{\partial}{\partial p}\left(\frac{H_qH_s}{H_p}\right)+u'_q\frac{\partial}{\partial q}\left(\frac{H_pH_s}{H_q}\right)+u'_s\frac{\partial}{\partial s}\left(\frac{H_pH_q}{H_s}\right)\right].$$

Пример №199 Дано скалярное поле в цилиндрических координатах:

$$u=r^2\varphi+z^2\varphi^3-r\varphi z \tag{23}$$

Найти Δu.

В цилиндрических координатах

$$\Delta u=\left[\frac{1}{H_p^2}u''_{pp}+\frac{1}{H_q^2}u''_{qq}+\frac{1}{H_s^2}u''_{ss}\right]+\frac{1}{H_pH_qH_s}\left[u'_p\frac{\partial}{\partial p}\left(\frac{H_qH_s}{H_p}\right)+u'_q\frac{\partial}{\partial q}\left(\frac{H_pH_s}{H_q}\right)+u'_s\frac{\partial}{\partial s}\left(\frac{H_pH_q}{H_s}\right)\right]=$$

$$=\left[\frac{1}{H_r^2}u''_{rr}+\frac{1}{H_\varphi^2}u''_{\varphi\varphi}+\frac{1}{H_z^2}u''_{zz}\right]+\frac{1}{H_rH_\varphi H_z}\left[u'_r\frac{\partial}{\partial r}\left(\frac{H_\varphi H_z}{H_r}\right)+u'_\varphi\frac{\partial}{\partial\varphi}\left(\frac{H_rH_z}{H_\varphi}\right)+u'_z\frac{\partial}{\partial z}\left(\frac{H_rH_\varphi}{H_z}\right)\right]=$$

$$=\left[u''_{rr}+\frac{1}{r^2}u''_{\varphi\varphi}+u''_{zz}\right]+\frac{1}{r}\left[u'_r\frac{\partial}{\partial r}r+u'_\varphi\frac{\partial}{\partial\varphi}\frac{1}{r}+u'_z\frac{\partial}{\partial z}r\right]= \tag{24}$$

$$=u''_{rr}+\frac{1}{r^2}u''_{\varphi\varphi}+u''_{zz}+\frac{1}{r}u'_r.$$

В нашем случае

$$u'_r=2r\varphi-\varphi z,\quad u''_{rr}=2\varphi, \tag{25}$$

$$u'_\varphi=r^2+3z^2\varphi^2-rz,\quad u''_{\varphi\varphi}=6z^2\varphi, \tag{26}$$

$$u'_z=2z\varphi^3-r\varphi,\quad u''_{zz}=2\varphi^3; \tag{27}$$

$$\Delta u=u''_{rr}+\frac{1}{r^2}u''_{\varphi\varphi}+u''_{zz}+\frac{1}{r}u'_r=2\varphi+\frac{1}{r^2}6z^2\varphi+2\varphi^3+\frac{1}{r}\left(2r\varphi-\varphi z\right)=$$

$$=2\varphi+6\frac{z^2\varphi}{r^2}+2\varphi^3+2\varphi-\frac{\varphi z}{r}=2\varphi^3+4\varphi-\frac{\varphi z}{r}+6\frac{z^2\varphi}{r^2}. \tag{28}$$