$$\frac{\partial}{\partial y}e^{-y\left(1+x^2\right)} = -\left(1+x^2\right)e^{-y\left(1+x^2\right)}, \tag{1}$$

$$\frac{\partial}{\partial y}\frac{e^{-y\left(1+x^2\right)}}{1+x^2} = -e^{-y\left(1+x^2\right)}, \tag{2}$$

$$\int\limits_0^\infty e^{-y\left(1+x^2\right)}dy = -\left.\frac{e^{-y\left(1+x^2\right)}}{1+x^2}\right|_0^\infty = -\left(0-\frac{1}{1+x^2}\right) = \frac{1}{1+x^2}. \tag{3}$$

Теперь перейдём к основному интегралу.

$$L\left(\alpha\right) = \int\limits_0^\infty \frac{\cos\alpha x}{1+x^2}dx = \int\limits_0^\infty \cos\alpha x\int\limits_0^\infty e^{-y\left(1+x^2\right)}dydx = \int\limits_0^\infty\int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dxdy \tag{4}$$

Введём интеграл с двумя параметрами

$$I\left(\alpha, y\right) = \int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dx, \tag{5}$$

$$I'_\alpha = -\int\limits_0^\infty e^{-y\left(1+x^2\right)}x\sin\alpha x\,dx, \tag{6}$$

$$\frac{\partial}{\partial x}e^{-y\left(1+x^2\right)} = -ye^{-y\left(1+x^2\right)}\frac{\partial}{\partial x}\left(1+x^2\right) = -2xye^{-y\left(1+x^2\right)}, \tag{7}$$

$$I'_\alpha = \frac{1}{2y}\int\limits_0^\infty\left(-2xye^{-y\left(1+x^2\right)}\right)\sin\alpha x\,dx = \frac{1}{2y}\int\limits_0^\infty\left(e^{-y\left(1+x^2\right)}\right)'\sin\alpha x\,dx = \left.\frac{1}{2y}e^{-y\left(1+x^2\right)}\sin\alpha x\right|_0^\infty -$$

$$-\frac{1}{2y}\int\limits_0^\infty\left(\sin\alpha x\right)'e^{-y\left(1+x^2\right)}dx = -\frac{\alpha}{2y}\int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dx = -\frac{\alpha}{2y}I. \tag{8}$$

Дифференциальное уравнение:

$$\frac{I'_\alpha}{I} = -\frac{\alpha}{2y}. \tag{9}$$

$$\ln|I| = -\frac{\alpha^2}{4y} + \tilde{\varphi}\left(y\right), \tag{10}$$

$$|I| = e^{-\frac{\alpha^2}{4y}+\tilde{\varphi}(y)} = e^{\tilde{\varphi}(y)}e^{-\frac{\alpha^2}{4y}}, \tag{11}$$

$$I = \pm e^{\tilde{\varphi}(y)}e^{-\frac{\alpha^2}{4y}} = \varphi\left(y\right)e^{-\frac{\alpha^2}{4y}}. \tag{12}$$

$\alpha = 0$:

$$I\left(0, y\right) = \varphi\left(y\right) = \int\limits_0^\infty e^{-y\left(1+x^2\right)}dx = e^{-y}\int\limits_0^\infty e^{-yx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-y}, \tag{13}$$

$$I = \varphi\left(y\right)e^{-\frac{\alpha^2}{4y}} = \frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-y}e^{-\frac{\alpha^2}{4y}} = \frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-\left(y+\frac{\alpha^2}{4y}\right)}. \tag{14}$$

Вернёмся к основной задаче.

$$L\left(\alpha\right) = \int\limits_0^\infty\int\limits_0^\infty e^{-y\left(1+x^2\right)}\cos\alpha x\,dxdy = \int\limits_0^\infty Idy = \int\limits_0^\infty \frac{1}{2}\sqrt{\frac{\pi}{y}}e^{-\left(y+\frac{\alpha^2}{4y}\right)}dy. \tag{15}$$

$\sqrt{y} = z$

$$L\left(\alpha\right) = \sqrt{\pi}\int\limits_0^\infty e^{-\left(y+\frac{\alpha^2}{4y}\right)}\frac{dy}{2\sqrt{y}} = \sqrt{\pi}\int\limits_0^\infty e^{-\left(z^2+\frac{\alpha^2}{4z^2}\right)}dz \tag{16}$$

Используем результаты №3807:

$$\int\limits_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)}\,dx = \frac{\sqrt{\pi}}{2}e^{-2\,|a|} \tag{17}$$

Заменим тут $a = \frac{\alpha}{2}$, $x = z$

$$\int\limits_0^\infty e^{-\left(z^2+\frac{\alpha^2}{4z^2}\right)}\,dz = \frac{\sqrt{\pi}}{2}e^{-|\alpha|} \tag{18}$$

и подставим в (16)

$$L\left(\alpha\right) = \sqrt{\pi}\int\limits_0^\infty e^{-\left(z^2+\frac{\alpha^2}{4z^2}\right)}dz = \frac{\pi}{2}e^{-|\alpha|} \tag{19}$$