1 Рассматриваемые интегралы

$$S_m(x,y)=\sum_{k=0}^{m}\alpha_{mk}x^{m-k}y^k \tag{1}$$

$$P_n(x,y)=\sum_{m=0}^{n}S_m(x,y) \tag{2}$$

$$R(x,y)=\frac{P_j(x,y)}{Q_n(x,y)} \tag{3}$$

$$\int R(\sin x,\cos x)\,dx \tag{4}$$

2 Универсальная подстановка

$$\operatorname{tg}\frac{x}{2}\equiv t \qquad \left(\frac{x}{2}=\operatorname{arctg}t\right),\quad 0\leqslant x\leqslant\pi \tag{5}$$

$$dx=\frac{2dt}{1+t^2} \tag{6}$$

$$\sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1 \quad\Rightarrow\quad \operatorname{tg}^2\frac{x}{2}+1=\frac{1}{\cos^2\frac{x}{2}} \tag{7}$$

$$\cos^2\frac{x}{2}=\frac{1}{\operatorname{tg}^2\frac{x}{2}+1}=\frac{1}{t^2+1} \tag{8}$$

$$\sin^2\frac{x}{2}=1-\cos^2\frac{x}{2}=1-\frac{1}{t^2+1}=\frac{t^2}{t^2+1} \tag{9}$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{1}{t^2+1}-\frac{t^2}{t^2+1}=\frac{1-t^2}{1+t^2} \tag{10}$$

$$\sin x=2\cos\frac{x}{2}\sin\frac{x}{2}=2\sqrt{\frac{1}{t^2+1}}\sqrt{\frac{t^2}{t^2+1}}=\frac{2t}{t^2+1} \tag{11}$$

$$\int R(\sin x,\cos x)\,dx=\int R\left(\frac{2t}{t^2+1},\frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2} \tag{12}$$

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$$\int\frac{\sin^2 x}{\sin x+2\cos x}dx=\int\frac{\left(\frac{2t}{t^2+1}\right)^2}{\frac{2t}{t^2+1}+2\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2}=\int\frac{\left(\frac{2t}{t^2+1}\right)^2}{t+1-t^2}dt=\int\frac{-4t^2}{(t^2-t-1)\left(t^2+1\right)^2}dt= \tag{13}$$

$$=\frac{S}{t^2+1}+\int\frac{T}{(t^2-t-1)(t^2+1)}dt=\frac{At+B}{t^2+1}+\int\left(\frac{Ct+D}{t^2-t-1}+\frac{Et+F}{t^2+1}\right)dt$$

Дальше знакомое:

$$\int\frac{-4t^2}{(t^2-t-1)\left(t^2+1\right)^2}dt=\frac{At+B}{t^2+1}+\int\left(\frac{Ct+D}{t^2-t-1}+\frac{Et+F}{t^2+1}\right)dt \tag{14}$$

$$\frac{-4t^2}{(t^2-t-1)\left(t^2+1\right)^2}=\frac{A\left(t^2+1\right)-2t\left(At+B\right)}{\left(t^2+1\right)^2}+\frac{Ct+D}{t^2-t-1}+\frac{Et+F}{t^2+1}\Bigg|\cdot\left(t^2-t-1\right)\left(t^2+1\right)^2 \tag{15}$$

$$-4t^2=\left[A\left(t^2+1\right)-2t\left(At+B\right)\right]\left(t^2-t-1\right)+\left(Ct+D\right)\left(t^2+1\right)^2+\left(Et+F\right)\left(t^2-t-1\right)\left(t^2+1\right) \tag{16}$$

$$\begin{cases} E+C=0\\ F-E+D-A=0\\ -F+2C-2B+A=0\\ -E+2D+2B+2A=-4\\ -F-E+C+2B-A=0\\ -F+D-A=0\end{cases}\quad\Longrightarrow\quad\begin{cases}A=-\frac{4}{5}\\ B=-\frac{2}{5}\\ C=0\\ D=-\frac{4}{5}\\ E=0\\ F=0\end{cases} \tag{17}$$

$$\int\frac{\sin^2 x}{\sin x+2\cos x}dx=\int\frac{-4t^2}{\left(t^2+1\right)^2\left(t^2-t-1\right)}dt=-\frac{2}{5}\frac{2t+1}{t^2+1}-\frac{4}{5}\int\frac{dt}{t^2-t-1} \tag{18}$$

$$t^2-t-1=\left(t^2-2t\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{4}-1=\left(t-\frac{1}{2}\right)^2-\frac{5}{4}=-\frac{5}{4}\left[1-\left(\frac{2t-1}{\sqrt{5}}\right)^2\right] \tag{19}$$

$\frac{2t-1}{\sqrt{5}} \equiv s,\ \frac{2dt}{\sqrt{5}} = ds,\ dt = \frac{\sqrt{5}}{2}ds$

$$\int \frac{dt}{t^2-t-1} = -\frac{4}{5}\int \frac{dt}{1-\left(\frac{2t-1}{\sqrt{5}}\right)^2} = -\frac{4}{5}\int \frac{\frac{\sqrt{5}}{2}ds}{1-s^2} = -\frac{2}{\sqrt{5}}\int \frac{ds}{1-s^2} = -\frac{2}{\sqrt{5}}\ln\left|\frac{1+s}{1-s}\right| + C_1 =$$

$$= -\frac{2}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-1+2t}{\sqrt{5}+1-2t}\right| + C_1 \tag{20}$$

$$\int \frac{\sin^2 x}{\sin x + 2\cos x}dx = -\frac{2}{5}\frac{2t+1}{t^2+1} - \frac{4}{5}\int \frac{dt}{t^2-t-1} = -\frac{2}{5}\frac{2t+1}{t^2+1} - \frac{4}{5}\left[-\frac{2}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-1+2t}{\sqrt{5}+1-2t}\right| + C_1\right] =$$

$-\frac{4}{5}C_1 \equiv C$

$$= -\frac{2}{5}\frac{2t+1}{t^2+1} + \frac{8}{5\sqrt{5}}\ln\left|\frac{\sqrt{5}-1+2t}{\sqrt{5}+1-2t}\right| - \frac{4}{5}C_1 = -\frac{2}{5}\frac{2\mathrm{tg}\,\frac{x}{2}+1}{\mathrm{tg}^2\,\frac{x}{2}+1} + \frac{8}{5\sqrt{5}}\ln\left|\frac{\sqrt{5}-1+2\mathrm{tg}\,\frac{x}{2}}{\sqrt{5}+1-2\mathrm{tg}\,\frac{x}{2}}\right| + C \tag{21}$$

3 Специальные случаи

1)

$$R(-\sin x, \cos x) = -R(\sin x, \cos x) \tag{22}$$

$$\cos x \equiv t \tag{23}$$

2)

$$R(\sin x, -\cos x) = -R(\sin x, \cos x) \tag{24}$$

$$\sin x \equiv t \tag{25}$$

3)

$$R(-\sin x, -\cos x) = R(\sin x, \cos x) \tag{26}$$

$$\mathrm{tg}\,x \equiv t \tag{27}$$

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$$\int \frac{dx}{(2+\cos x)\sin x} \tag{28}$$

$$\frac{1}{(2+\cos x)(-\sin x)} = -\frac{1}{(2+\cos x)\sin x} \quad \Rightarrow \quad \cos x = t, \quad -\sin x dx = dt \tag{29}$$

$$\int \frac{dx}{(2+\cos x)\sin x} = -\int \frac{-\sin x dx}{(2+\cos x)\sin^2 x} = -\int \frac{-\sin x dx}{(2+\cos x)(1-\cos^2 x)} = -\int \frac{dt}{(2+t)(1-t^2)} \tag{30}$$

$$\frac{1}{(2+t)(1-t^2)} = \frac{1}{(2+t)(1-t)(1+t)} = \frac{A}{2+t} + \frac{B}{1-t} + \frac{C}{1+t} \tag{31}$$

$$\begin{cases} -C+B-A=0 \\ 3B-C=0 \\ 2C+2B+A=1 \end{cases} \Longrightarrow \begin{cases} A=-\frac{1}{3} \\ B=\frac{1}{6} \\ C=\frac{1}{2} \end{cases} \tag{32}$$

$$\frac{1}{(2+t)(1-t^2)} = -\frac{1}{3}\frac{1}{2+t} + \frac{1}{6}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t} \tag{33}$$

$$\int \frac{dt}{(2+t)(1-t^2)} = \int \left(-\frac{1}{3}\frac{1}{2+t} + \frac{1}{6}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t}\right)dt = -\frac{1}{3}\ln|2+t| - \frac{1}{6}\ln|1-t| + \frac{1}{2}\ln|1+t| + C =$$

$$= -\frac{1}{3}\ln|2+\cos x| - \frac{1}{6}\ln|1-\cos x| + \frac{1}{2}\ln|1+\cos x| + C \tag{34}$$

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$$\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} \tag{35}$$

$$\frac{1}{a^2(-\sin x)^2 + b^2(-\cos x)^2} = \frac{1}{a^2\sin^2 x + b^2\cos^2 x} \quad \Rightarrow \quad \mathrm{tg}\,x \equiv t, \quad \frac{dx}{\cos^2 x} = dt \tag{36}$$

$$\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \int \frac{1}{a^2\mathrm{tg}^2 x + b^2}\frac{dx}{\cos^2 x} = \int \frac{1}{a^2t^2+b^2}dt = \frac{1}{b^2}\int \frac{1}{\left(\frac{at}{b}\right)^2+1}dt =$$

$$= \frac{1}{ab}\mathrm{arctg}\,\frac{at}{b} + C = \frac{1}{ab}\mathrm{arctg}\left(\frac{a}{b}\mathrm{tg}\,x\right) + C \tag{37}$$