Преобразование градиента, дивергенции и (если повезёт) ротора в криволинейные координаты.

1 Общая часть

Замена $(x, y, z) \longrightarrow (p, q, s)$

$$\left\{ \begin{array}{l} x = x\,(p, q, s) \\ y = y\,(p, q, s) \\ z = z\,(p, q, s) \end{array} \right. \tag{1}$$

Радиус-вектор $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$

$$\vec{r}'_p = \vec{i}x'_p + \vec{j}y'_p + \vec{k}z'_p \tag{2}$$

Коэффициенты Ламэ:

$$H_p \equiv |\vec{r}'_p| = \sqrt{\left(x'_p\right)^2 + \left(y'_p\right)^2 + \left(z'_p\right)^2} \tag{3}$$

$$\vec{e}_p \equiv \frac{\vec{r}'_p}{H_p}, \qquad |\vec{e}_p| = 1 \tag{4}$$

Аналогично,

$$H_q \equiv \left|\vec{r}'_q\right|, \quad H_s \equiv |\vec{r}'_s| \tag{5}$$

$$\vec{e}_q \equiv \frac{\vec{r}'_q}{H_q}, \quad \vec{e}_s \equiv \frac{\vec{r}'_s}{H_s} \tag{6}$$

Векторы $\vec{e}_p, \vec{e}_q, \vec{e}_s$ – орты новой системы координат. У *ортогональных* систем они имеют такие свойства:

$$(\vec{e}_p \cdot \vec{e}_p) = 1, \quad (\vec{e}_q \cdot \vec{e}_q) = 1, \quad (\vec{e}_s \cdot \vec{e}_s) = 1, \tag{7}$$

$$(\vec{e}_p \cdot \vec{e}_q) = 0, \quad (\vec{e}_p \cdot \vec{e}_s) = 0, \quad (\vec{e}_q \cdot \vec{e}_s) = 0, \tag{8}$$

$$[\vec{e}_p \times \vec{e}_q] = \vec{e}_s, \quad [\vec{e}_q \times \vec{e}_s] = \vec{e}_p, \quad [\vec{e}_s \times \vec{e}_p] = \vec{e}_q. \tag{9}$$

Продифференцируем (1) по x, y и z:

$$\left\{ \begin{array}{l} 1 = x'_p p'_x + x'_q q'_x + x'_s s'_x \\ 0 = y'_p p'_x + y'_q q'_x + y'_s s'_x \\ 0 = z'_p p'_x + z'_q q'_x + z'_s s'_x \end{array} \right. \qquad \left\{ \begin{array}{l} 0 = x'_p p'_y + x'_q q'_y + x'_s s'_y \\ 1 = y'_p p'_y + y'_q q'_y + y'_s s'_y \\ 0 = z'_p p'_y + z'_q q'_y + z'_s s'_y \end{array} \right. \qquad \left\{ \begin{array}{l} 0 = x'_p p'_z + x'_q q'_z + x'_s s'_z \\ 0 = y'_p p'_z + y'_q q'_z + y'_s s'_z \\ 1 = z'_p p'_z + z'_q q'_z + z'_s s'_z \end{array} \right. \tag{10}$$

По правилу Крамера, из первой системы

$$p_x = \frac{\begin{vmatrix} 1 & x'_q & x'_s \\ 0 & y'_q & y'_s \\ 0 & z'_q & z'_s \end{vmatrix}}{\begin{vmatrix} x'_p & x'_q & x'_s \\ y'_p & y'_q & y'_s \\ z'_p & z'_q & z'_s \end{vmatrix}} = \frac{\left(\vec{i}, \vec{r}'_q, \vec{r}'_s\right)}{\left(\vec{r}'_p, \vec{r}'_q, \vec{r}'_s\right)} = \frac{H_q H_s \left(\vec{i}, \vec{e}_q, \vec{e}_s\right)}{H_p H_q H_s \left(\vec{e}_p, \vec{e}_q, \vec{e}_s\right)} = \frac{1}{H_p} \frac{\vec{i} \cdot [\vec{e}_q \times \vec{e}_s]}{\vec{e}_p \cdot [\vec{e}_q \times \vec{e}_s]} = \frac{1}{H_p} \frac{\vec{i} \cdot \vec{e}_p}{\vec{e}_p \cdot \vec{e}_p} = \frac{\vec{i} \cdot \vec{e}_p}{H_p}. \tag{11}$$

Аналогично, из второй системы в (10) получаем

$$p_y = \frac{\begin{vmatrix} 0 & x'_q & x'_s \\ 1 & y'_q & y'_s \\ 0 & z'_q & z'_s \end{vmatrix}}{\begin{vmatrix} x'_p & x'_q & x'_s \\ y'_p & y'_q & y'_s \\ z'_p & z'_q & z'_s \end{vmatrix}} = \frac{\vec{j} \cdot \vec{e}_p}{H_p}, \qquad p_z = \frac{\begin{vmatrix} 0 & x'_q & x'_s \\ 0 & y'_q & y'_s \\ 1 & z'_q & z'_s \end{vmatrix}}{\begin{vmatrix} x'_p & x'_q & x'_s \\ y'_p & y'_q & y'_s \\ z'_p & z'_q & z'_s \end{vmatrix}} = \frac{\vec{k} \cdot \vec{e}_p}{H_p}. \tag{12}$$

Пример Циллиндрические координаты:

$$\vec{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ h \end{pmatrix} \tag{13}$$

$$\vec{r}'_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \qquad \vec{r}'_\varphi = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}, \qquad \vec{r}'_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

$$H_r = \sqrt{\cos^2\varphi + \sin^2\varphi + 0^2} = 1 \tag{15}$$

$$H_\varphi = \sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi + 0^2} = \sqrt{r^2} = r \tag{16}$$

$$H_h = 1 \tag{17}$$

$$\vec{e}_r = \frac{\vec{r}'_r}{H_r} = \vec{r}'_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \qquad \vec{e}_\varphi = \frac{\vec{r}'_\varphi}{H_\varphi} = \frac{\vec{r}'_\varphi}{r} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \qquad \vec{e}_h = \frac{\vec{r}'_h}{H_h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{18}$$

Ортогональность:

$$\vec{e}_r \cdot \vec{e}_h = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \qquad \vec{e}_\varphi \cdot \vec{e}_h = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \tag{19}$$

$$\vec{e}_r \cdot \vec{e}_\varphi = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi = 0 \tag{20}$$

Правота:

$$\vec{e}_r \times \vec{e}_\varphi = \begin{pmatrix} \vec{i} & \cos\varphi & -\sin\varphi \\ \vec{j} & \sin\varphi & \cos\varphi \\ \vec{k} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi\cdot 0 + \sin\varphi\cdot 0 \\ -\sin\varphi\cdot 0 - \cos\varphi\cdot 0 \\ \cos^2\varphi + \sin^2\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{e}'_h \tag{21}$$

$$\vec{e}_\varphi \times \vec{e}_h = \vec{e}_\varphi \times [\vec{e}_r \times \vec{e}_\varphi] = \vec{e}_r(\vec{e}_\varphi \cdot \vec{e}_\varphi) - \vec{e}_\varphi(\vec{e}_\varphi \cdot \vec{e}_r) = \vec{e}_r \cdot 1 - \vec{e}_\varphi \cdot 0 = \vec{e}_r \tag{22}$$

$$\vec{e}_h \times \vec{e}_r = \vec{e}_h \times [\vec{e}_\varphi \times \vec{e}_h] = \vec{e}_\varphi(\vec{e}_h \cdot \vec{e}_h) - \vec{e}_h(\vec{e}_h \cdot \vec{e}_\varphi) = \vec{e}_\varphi \tag{23}$$

Задание: получить орты, к. Ламэ и проверить ортогональность для сферических координат.

2 Градиент

Так как для всякого вектора $\vec{a} = \vec{i}a_x + \vec{j}a_y + \vec{k}a_z = \vec{i}\left(\vec{i}\cdot\vec{a}\right) + \vec{j}\left(\vec{j}\cdot\vec{a}\right) + \vec{k}\left(\vec{k}\cdot\vec{a}\right)$,

$$\operatorname{grad} p = \vec{i}p'_x + \vec{j}p'_y + \vec{k}p'_z = \vec{i}\frac{\vec{i}\cdot\vec{e}_p}{H_p} + \vec{j}\frac{\vec{j}\cdot\vec{e}_p}{H_p} + \vec{k}\frac{\vec{k}\cdot\vec{e}_p}{H_p} = \frac{\vec{e}_p}{H_p}. \tag{24}$$

Решая системы (10) относительно производных других координат, тем же путём находим, что

$$\operatorname{grad} q = \frac{\vec{e}_q}{H_q}, \quad \operatorname{grad} s = \frac{\vec{e}_s}{H_s}. \tag{25}$$

Градиент произвольной функции $u(p,q,s)$ находится отсюда мгновенно:

$$\operatorname{grad} u = u'_p \operatorname{grad} p + u'_q \operatorname{grad} q + u'_s \operatorname{grad} s = \frac{u'_p}{H_p}\vec{e}_p + \frac{u'_q}{H_q}\vec{e}_q + \frac{u'_s}{H_s}\vec{e}_s \tag{26}$$

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$$u = r^2 + 2r\cos\varphi - e^z\sin\varphi \tag{27}$$

$$\operatorname{grad} u = \frac{u'_r}{H_r}\vec{e}_r + \frac{u'_\varphi}{H_\varphi}\vec{e}_\varphi + \frac{u'_h}{H_z}\vec{e}_z = u'_r\vec{e}_r + \frac{u'_\varphi}{r}\vec{e}_\varphi + u'_z\vec{e}_z =$$

$$= \frac{\partial}{\partial r}\left(r^2 + 2r\cos\varphi - e^z\sin\varphi\right)\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\varphi}\left(r^2 + 2r\cos\varphi - e^z\sin\varphi\right)\vec{e}_\varphi + \frac{\partial}{\partial z}\left(r^2 + 2r\cos\varphi - e^z\sin\varphi\right)\vec{e}_z =$$

$$= (2r + 2\cos\varphi)\vec{e}_r + \frac{1}{r}(-2r\sin\varphi - e^z\cos\varphi)\vec{e}_\varphi + (-e^z\sin\varphi)\vec{e}_z =$$

$$= 2(r + \cos\varphi)\vec{e}_r - \left(2\sin\varphi + \frac{e^z}{r}\cos\varphi\right)\vec{e}_\varphi - e^z\sin\varphi\vec{e}_z \tag{28}$$

3 Дифференциальные свойства ортов

$$\frac{\partial}{\partial\gamma}\left|\vec{e}_\alpha\cdot\vec{e}_\beta=\left\{\begin{array}{ll}1, & \alpha=\beta\\ 0, & \alpha\neq\beta\end{array}\right.\right. \tag{29}$$

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha+\vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\beta=0 \tag{30}$$

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha=-\vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\beta \tag{31}$$

где α, β и γ – произвольные координаты из новой системы координат. В частности, если $\beta=\alpha$, то

$$\vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha=-\vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha, \tag{32}$$

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$$\vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha=0. \tag{33}$$

Далее рассмотрим равенство

$$\vec{r}''_{\alpha\beta}=\vec{r}''_{\beta\alpha} \tag{34}$$

$$\frac{\partial}{\partial\beta}\left(H_\alpha\vec{e}_\alpha\right)=\frac{\partial}{\partial\alpha}\left(H_\beta\vec{e}_\beta\right) \tag{35}$$

$$\left.\frac{\partial}{\partial\beta}H_\alpha\vec{e}_\alpha+H_\alpha\frac{\partial}{\partial\beta}\vec{e}_\alpha=\frac{\partial}{\partial\alpha}H_\beta\vec{e}_\beta+H_\beta\frac{\partial}{\partial\alpha}\vec{e}_\beta\right|\cdot\vec{e}_\beta \tag{36}$$

В силу (33)

$$H_\alpha\vec{e}_\beta\cdot\frac{\partial}{\partial\beta}\vec{e}_\alpha=\frac{\partial}{\partial\alpha}H_\beta, \tag{37}$$

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\beta}\vec{e}_\alpha=\frac{1}{H_\alpha}\frac{\partial}{\partial\alpha}H_\beta, \tag{38}$$

где α и β – две произвольные координаты из новой системы координат ($\alpha\neq\beta$). Теперь умножим

$$\left.\frac{\partial}{\partial\beta}H_\alpha\vec{e}_\alpha+H_\alpha\frac{\partial}{\partial\beta}\vec{e}_\alpha=\frac{\partial}{\partial\alpha}H_\beta\vec{e}_\beta+H_\beta\frac{\partial}{\partial\alpha}\vec{e}_\beta\right|\cdot\vec{e}_\gamma \tag{39}$$

($\gamma\neq\alpha$, $\gamma\neq\beta$)

$$H_\alpha\vec{e}_\gamma\cdot\frac{\partial}{\partial\beta}\vec{e}_\alpha=H_\beta\vec{e}_\gamma\cdot\frac{\partial}{\partial\alpha}\vec{e}_\beta \tag{40}$$

$$\vec{e}_\gamma\cdot\frac{\partial}{\partial\beta}\vec{e}_\alpha=\frac{H_\beta}{H_\alpha}\vec{e}_\gamma\cdot\frac{\partial}{\partial\alpha}\vec{e}_\beta \tag{41}$$

4 Дивергенция

Для поля $\vec{F}=\vec{i}P+\vec{j}Q+\vec{k}R$ дивергенция примет вид

$$\begin{aligned}\operatorname{div}\vec{F}&=P'_x+Q'_y+R'_z=\\&=\left(P'_pp'_x+P'_qq'_x+P'_ss'_x\right)+\left(Q'_pp'_y+Q'_qq'_y+Q'_ss'_y\right)+\left(R'_pp'_z+R'_qq'_z+R'_ss'_z\right)=\\&=\left(P'_pp'_x+Q'_pp'_y+R'_pp'_z\right)+\left(P'_qq'_x+Q'_qq'_y+R'_qq'_z\right)+\left(P'_ss'_x+Q'_ss'_y+R'_ss'_z\right)=\\&=\vec{F}'_p\cdot\operatorname{grad}p+\vec{F}'_q\cdot\operatorname{grad}q+\vec{F}'_s\cdot\operatorname{grad}s=\vec{F}'_p\cdot\frac{\vec{e}_p}{H_p}+\vec{F}'_q\cdot\frac{\vec{e}_q}{H_q}+\vec{F}'_s\cdot\frac{\vec{e}_s}{H_s}\end{aligned} \tag{42}$$

Обозначим $\vec{F}=F_p\vec{e}_p+F_q\vec{e}_q+F_s\vec{e}_s$:

$$\vec{F}'_p\cdot\vec{e}_p=\frac{\partial}{\partial p}\left(F_p\vec{e}_p+F_q\vec{e}_q+F_s\vec{e}_s\right)\cdot\vec{e}_p=$$

$$=\left(\frac{\partial}{\partial p}F_p\vec{e}_p+\frac{\partial}{\partial p}F_q\vec{e}_q+\frac{\partial}{\partial p}F_s\vec{e}_s+F_p\frac{\partial}{\partial p}\vec{e}_p+F_q\frac{\partial}{\partial p}\vec{e}_q+F_s\frac{\partial}{\partial p}\vec{e}_s\right)\cdot\vec{e}_p=$$

по (38)

$$=\frac{\partial}{\partial p}F_p+F_q\vec{e}_p\cdot\frac{\partial}{\partial p}\vec{e}_q+F_s\vec{e}_p\cdot\frac{\partial}{\partial p}\vec{e}_s=\frac{\partial}{\partial p}F_p+F_q\frac{1}{H_q}\frac{\partial}{\partial q}H_p+F_s\frac{1}{H_s}\frac{\partial}{\partial s}H_p= \tag{43}$$

$$= \frac{1}{H_q H_s}\left(H_q H_s \frac{\partial}{\partial p} F_p + F_q H_s \frac{\partial}{\partial q} H_p + F_s H_q \frac{\partial}{\partial s} H_p\right).$$

Так же получим, что

$$\vec{F}'_q \cdot \vec{e}_q = \frac{1}{H_p H_s}\left(H_p H_s \frac{\partial}{\partial q} F_q + F_p H_s \frac{\partial}{\partial p} H_q + F_s H_p \frac{\partial}{\partial s} H_q\right), \tag{44}$$

$$\vec{F}'_s \cdot \vec{e}_s = \frac{1}{H_p H_q}\left(H_p H_q \frac{\partial}{\partial s} F_s + F_p H_q \frac{\partial}{\partial p} H_s + F_q H_p \frac{\partial}{\partial q} H_s\right). \tag{45}$$

Тогда

$$\operatorname{div} \vec{F} = \vec{F}'_p \cdot \frac{\vec{e}_p}{H_p} + \vec{F}'_q \cdot \frac{\vec{e}_q}{H_q} + \vec{F}'_s \cdot \frac{\vec{e}_s}{H_s} =$$

$$= \frac{1}{H_q H_s H_p}\left(H_q H_s \frac{\partial}{\partial p} F_p + F_q H_s \frac{\partial}{\partial q} H_p + F_s H_q \frac{\partial}{\partial s} H_p\right) +$$

$$\frac{1}{H_p H_s H_q}\left(H_p H_s \frac{\partial}{\partial q} F_q + F_p H_s \frac{\partial}{\partial p} H_q + F_s H_p \frac{\partial}{\partial s} H_q\right) +$$

$$\frac{1}{H_p H_q H_s}\left(H_p H_q \frac{\partial}{\partial s} F_s + F_p H_q \frac{\partial}{\partial p} H_s + F_q H_p \frac{\partial}{\partial q} H_s\right) = \tag{46}$$

Соберём одинаковые производные:

$$= \frac{1}{H_p H_q H_s}\left(H_q H_s \frac{\partial}{\partial p} F_p + F_p H_s \frac{\partial}{\partial p} H_q + F_p H_q \frac{\partial}{\partial p} H_s + \right.$$

$$+ H_p H_s \frac{\partial}{\partial q} F_q + F_q H_s \frac{\partial}{\partial q} H_p + F_q H_p \frac{\partial}{\partial q} H_s +$$

$$\left. H_p H_q \frac{\partial}{\partial s} F_s + F_s H_q \frac{\partial}{\partial s} H_p + F_s H_p \frac{\partial}{\partial s} H_q\right) =$$

$$= \frac{1}{H_p H_q H_s}\left[\frac{\partial}{\partial p}\left(H_q H_s F_p\right) + \frac{\partial}{\partial q}\left(H_p H_s F_q\right) + \frac{\partial}{\partial s}\left(H_p H_q F_s\right)\right]$$

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$$\vec{F} = \varphi \operatorname{arctg} r \vec{e}_r + 2\vec{e}_\varphi - z^2 e^z \vec{e}_z \tag{47}$$

$$\operatorname{div} \vec{F} = \frac{1}{H_r H_\varphi H_z}\left[\frac{\partial}{\partial r}\left(H_\varphi H_z F_r\right) + \frac{\partial}{\partial \varphi}\left(H_r H_z F_\varphi\right) + \frac{\partial}{\partial z}\left(H_r H_\varphi F_z\right)\right] =$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r F_r\right) + \frac{\partial}{\partial \varphi} F_\varphi + \frac{\partial}{\partial z}\left(r F_z\right)\right] = \tag{48}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\varphi \operatorname{arctg} r\right) + \frac{\partial}{\partial \varphi} 2 + \frac{\partial}{\partial z}\left(-z^2 e^z r\right)\right] = \frac{1}{r}\left[\varphi\left(\operatorname{arctg} r + \frac{r}{1+r^2}\right) - r\left(2ze^z + z^2 e^z\right)\right] =$$

$$= \frac{\varphi}{r} \operatorname{arctg} r + \frac{\varphi}{1+r^2} - (2+z)\, z e^z.$$