

1 Полезные интегралы

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)], \quad (1)$$

$k, n \in \mathbb{Z}; k, n \geq 0$:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos kx \cos nx dx &= \int_{-\pi}^{\pi} \frac{1}{2} [\cos (k+n)x + \cos (k-n)x] dx = \\ &= \frac{1}{2} \left[\frac{\sin (k+n)x}{k+n} + \frac{\sin (k-n)x}{k-n} \right] \Big|_{-\pi}^{\pi} = \frac{1}{2} \left[\frac{\sin (k+n)\pi}{k+n} + \frac{\sin (k-n)\pi}{k-n} + \frac{\sin (k+n)\pi}{k+n} + \frac{\sin (k-n)\pi}{k-n} \right] = 0, \end{aligned} \quad (2)$$

$k+n \neq 0, k-n \neq 0$.

- 1) $k+n \neq 0, k-n \neq 0$ (уже рассмотрели),
- 2) $k+n = 0, k-n \neq 0$,
- 3) $k+n \neq 0, k-n = 0$,
- 4) $k+n = 0, k-n = 0$.

Пусть $k+n = 0, k = -n$. $k, n \geq 0 \implies k = n = 0, k-n = 0$. Значит, мы рассматриваем четвёртый случай, а второй невозможен.

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \int_{-\pi}^{\pi} 1 \cdot 1 dx = 2\pi. \quad (3)$$

Третий случай, $k = n \neq 0$:

$$\begin{aligned} \int_{-\pi}^{\pi} \cos kx \cos nx dx &= \int_{-\pi}^{\pi} \cos^2 kx dx = \int_{-\pi}^{\pi} \frac{1 + \cos (2kx)}{2} dx = \\ &= \left[\frac{x}{2} + \frac{\cos (2kx)}{4k} \right] \Big|_{-\pi}^{\pi} = \pi + \frac{\cos (2k\pi)}{4k} - \frac{\cos (2k\pi)}{4k} = \pi. \end{aligned} \quad (4)$$

Итак,

$$\int_{-\pi}^{\pi} \cos kx \cos nx dx = \begin{cases} 0, & k \neq n, \\ \pi, & k = n \neq 0, \\ 2\pi, & k = n = 0. \end{cases} \quad (5)$$

Задание: аналогично получить всевозможные результаты для интегралов

$$\int_{-\pi}^{\pi} \sin kx \sin nx dx \quad \text{и} \quad \int_{-\pi}^{\pi} \sin kx \cos nx dx. \quad (6)$$

2 Ряд Фурье на одном обороте

Пусть на $x \in [-\pi, \pi]$

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (7)$$

(ряд Фурье).

$$f(x) \cos kx = \sum_{n=0}^{\infty} (a_n \cos nx \cos kx + b_n \sin nx \cos kx) \quad (8)$$

$k > 0$:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos kx dx &= \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} (a_n \cos nx \cos kx + b_n \sin nx \cos kx) dx = \\ &= \sum_{n=0}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos kx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos kx dx \right) = \sum_{n=0}^{\infty} (a_n \pi \delta_{nk} + b_n \cdot 0) = \pi a_k. \end{aligned} \quad (9)$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx. \quad (10)$$

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right) = 2\pi a_0, \quad (11)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \quad (12)$$

(От обозначений в Демидовиче мои отличаются расположением $\frac{1}{2}$ при a_0).

$$f(x) \sin kx = \sum_{n=0}^{\infty} (a_n \cos nx \sin kx + b_n \sin nx \sin kx), \quad (13)$$

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin kx dx &= \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} (a_n \cos nx \sin kx + b_n \sin nx \sin kx) dx = \\ &= \sum_{n=0}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \sin kx dx + b_n \int_{-\pi}^{\pi} \sin nx \sin kx dx \right) = \sum_{n=0}^{\infty} (a_n \cdot 0 + b_n \pi \delta_{nk}) = b_k \pi, \end{aligned} \quad (14)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx. \quad (15)$$

заменим обратно:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny dy, \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) dy, \quad (16)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny dy. \quad (17)$$

Задание: понять, почему мы не ищем b_0 .

3 Ряд Фурье на симметричном отрезке

Пусть $x \in [-l, l]$ Введём

$$y = \frac{\pi x}{l}, \quad x = \frac{ly}{\pi}, \quad (18)$$

$$g(y) = f(x) = f\left(\frac{ly}{\pi}\right). \quad (19)$$

$$g(y) = \sum_{n=0}^{\infty} (a_n \cos ny + b_n \sin ny), \quad (20)$$

где

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(y) \cos ny dy, \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) dy, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(y) \sin ny dy. \quad (21)$$

Обратно:

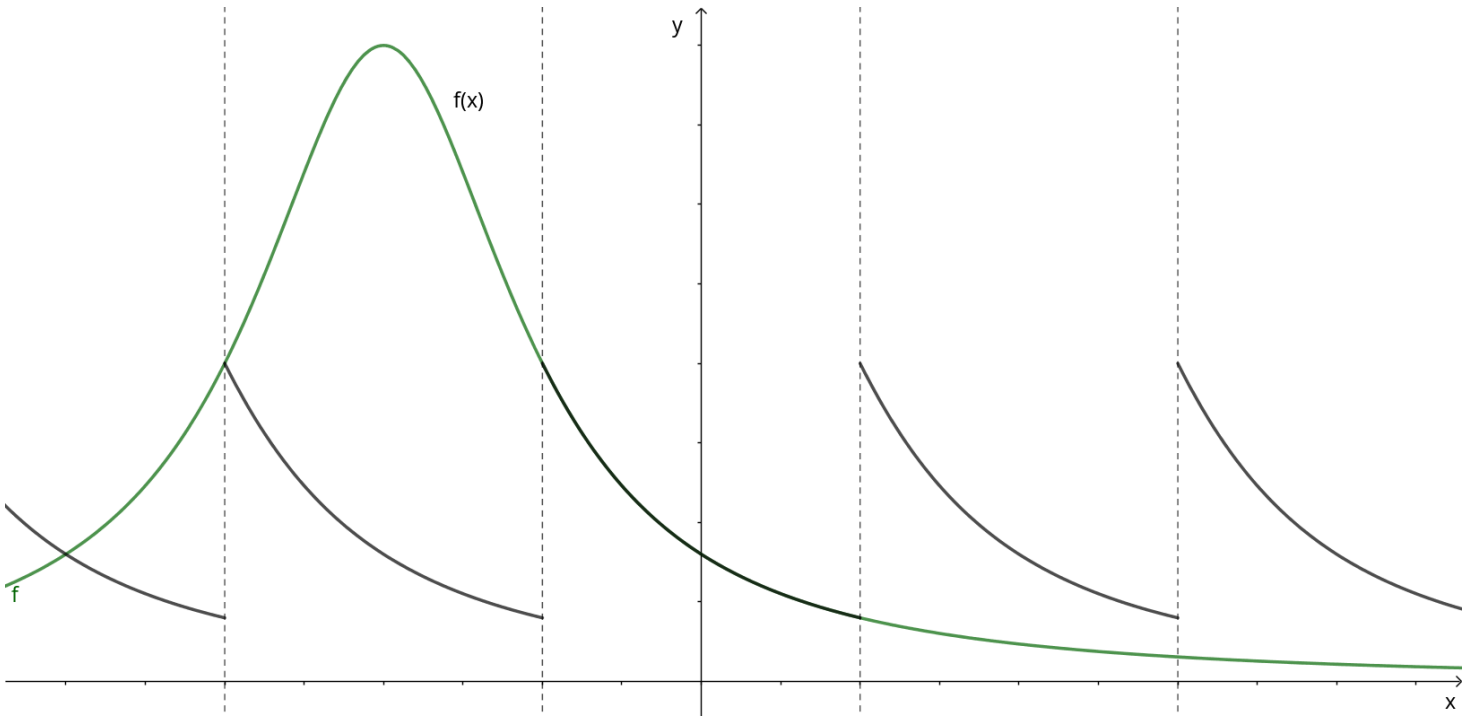
$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{\pi nx}{l} + b_n \sin \frac{\pi nx}{l} \right), \quad (22)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(y) \cos ny dy = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n \frac{\pi x}{l} d \frac{\pi x}{l} = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{\pi nx}{l} dx, \quad (23)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) d \frac{\pi x}{l} = \frac{1}{2l} \int_{-l}^l f(x) dx, \quad (24)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(y) \sin ny dy = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n \frac{\pi x}{l} d \frac{\pi x}{l} = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{\pi nx}{l} dx. \quad (25)$$

4 Периодичность ряда



5 Чётность

Пусть $\varphi(-x) = -\varphi(x)$. Тогда

$$\int_{-a}^a \varphi(x) dx = \int_{-a}^0 \varphi(x) dx + \int_0^a \varphi(x) dx = \quad (26)$$

$x = -y; x = y$

$$= -\int_a^0 \varphi(-y) dy + \int_0^a \varphi(y) dy = -\int_0^a \varphi(y) dy + \int_0^a \varphi(y) dy = 0.$$

Пусть теперь $\psi(-x) = \psi(x)$. Тогда

$$\begin{aligned} \int_{-a}^a \psi(x) dx &= \int_{-a}^0 \psi(x) dx + \int_0^a \psi(x) dx = \\ &= -\int_a^0 \psi(-y) dy + \int_0^a \psi(y) dy = \int_0^a \psi(y) dy + \int_0^a \psi(y) dy = 2 \int_0^a \psi(y) dy. \end{aligned} \quad (27)$$

Для чётной функции $f(x)$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{\pi n x}{l} dx = \frac{2}{l} \int_0^l f(x) \cos \frac{\pi n x}{l} dx, \quad (28)$$

$$a_0 = \frac{1}{2l} \int_{-l}^l f(x) dx = \frac{1}{l} \int_0^l f(x) dx, \quad (29)$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{\pi n x}{l} dx = 0. \quad (30)$$

Если же функция $f(x)$ нечётна,

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{\pi n x}{l} dx = 0, \quad (31)$$

$$a_0 = \frac{1}{2l} \int_{-l}^l f(x) dx = 0, \quad (32)$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{\pi nx}{l} dx = \frac{2}{l} \int_0^l f(x) \sin \frac{\pi nx}{l} dx. \quad (33)$$

5.1 Пример: №2944

Разложить в ряд Фурье в интервале $[-\pi, \pi]$ функцию $f(x) = \pi^2 - x^2$.

$l = \pi$, $f(x)$ чётна:

$$f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x), \quad (34)$$

поэтому $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) dx = \frac{1}{2\pi} \left(\pi^2 x - \frac{x^3}{3} \right) \Big|_{-\pi}^{\pi} = \frac{1}{2\pi} \left[\left(\pi^2 \pi - \frac{\pi^3}{3} \right) - \left(\pi^2 (-\pi) - \frac{(-\pi)^3}{3} \right) \right] = \frac{2}{3} \pi^2 \quad (35)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) \cos nx dx = \frac{1}{\pi n} \int_{-\pi}^{\pi} (\pi^2 - x^2) \sin' nx dx =$$

$$= \frac{1}{\pi n} (\pi^2 - x^2) \sin nx \Big|_{-\pi}^{\pi} - \frac{1}{\pi n} \int_{-\pi}^{\pi} (\pi^2 - x^2)' \sin nx dx = \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin nx dx = -\frac{2}{\pi n^2} \int_{-\pi}^{\pi} x \cos' nx dx = \quad (36)$$

$$= -\frac{2}{\pi n^2} x \cos nx \Big|_{-\pi}^{\pi} + \frac{2}{\pi n^2} \int_{-\pi}^{\pi} x' \cos nx dx = -\frac{2}{\pi n^2} x \cos nx \Big|_{-\pi}^{\pi} + \frac{2}{\pi n^3} \sin nx \Big|_{-\pi}^{\pi} = -\frac{2}{\pi n^2} [\pi \cos n\pi - (-\pi) \cos n(-\pi)] =$$

$$= -\frac{2}{\pi n^2} [\pi - (-\pi)] \cos n\pi = (-1)^{n+1} \frac{4}{n^2}.$$

С такими коэффициентами

$$f(x) = \pi^2 - x^2 = \sum_{n=0}^{\infty} \left(a_n \cos \frac{\pi nx}{l} + b_n \sin \frac{\pi nx}{l} \right) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} 4 \frac{(-1)^{n+1}}{n^2} \cos nx. \quad (37)$$

Кстати, при $x = \pi$: $\cos nx = (-1)^n$,

$$f(\pi) = \pi^2 - \pi^2 = 0 = \frac{2}{3} \pi^2 + \sum_{n=1}^{\infty} 4 \frac{(-1)^{n+1}}{n^2} (-1)^n = \frac{2}{3} \pi^2 - 4 \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad (38)$$

$$4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3} \pi^2, \quad (39)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (40)$$

Задание: решить самостоятельно № 2940, 2942, 2948, 2951.