

1 Проверка линейной зависимости

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = 0 \quad (1)$$

$$\alpha_1 y'_1 + \alpha_2 y'_2 + \dots + \alpha_n y'_n = 0 \quad (2)$$

$$\begin{cases} \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = 0 \\ \alpha_1 y'_1 + \alpha_2 y'_2 + \dots + \alpha_n y'_n = 0 \\ \vdots \\ \alpha_1 y_1^{(n-1)} + \alpha_2 y_2^{(n-1)} + \dots + \alpha_n y_n^{(n-1)} = 0 \end{cases} \quad (3)$$

$$\alpha_1 \begin{pmatrix} y_1 \\ y'_1 \\ \vdots \\ y_1^{(n-1)} \end{pmatrix} + \alpha_2 \begin{pmatrix} y_2 \\ y'_2 \\ \vdots \\ y_2^{(n-1)} \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} y_n \\ y'_n \\ \vdots \\ y_n^{(n-1)} \end{pmatrix} = 0 \quad (4)$$

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0 \quad (5)$$

(определитель Вронского)

$$641) y_1 = x + 2, y_2 = x - 2$$

$$\begin{vmatrix} x+2 & x-2 \\ 1 & 1 \end{vmatrix} = x+2 - (x-2) = 4 \neq 0 \quad (6)$$

независимы

$$642) y_1 = 6x + 9, y_2 = 8x + 12$$

$$\begin{vmatrix} 6x+9 & 8x+12 \\ 6 & 8 \end{vmatrix} = 48x+72 - 48x-72 = 0 \quad (7)$$

зависимы

$$644) y_1 = 1, y_2 = x, y_3 = x^2$$

$$\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

независимы

$$(643, 645 - 648)$$

2 Построение уравнений по решениям

$$y = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n \quad (8)$$

$\Downarrow$

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n & y \\ y'_1 & y'_2 & \cdots & y'_n & y' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} & y^{(n-1)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} & y^{(n)} \end{vmatrix} = 0 \quad (9)$$

$$A_n^n y^{(n)} + A_n^{n-1} y^{(n-1)} + \dots + A_n^2 y' + A_n^1 y = 0 \quad (10)$$

$$675) y_1 = x, y_2 = e^x$$

$$\begin{vmatrix} x & e^x & y \\ 1 & e^x & y' \\ 0 & e^x & y'' \end{vmatrix} = \begin{vmatrix} x & 1 & y \\ 1 & 1 & y' \\ 0 & 1 & y'' \end{vmatrix} e^x = \begin{vmatrix} x & 0 & y-y'' \\ 1 & 0 & y'-y'' \\ 0 & 1 & y'' \end{vmatrix} e^x = -e^x \begin{vmatrix} x & y-y'' \\ 1 & y'-y'' \end{vmatrix} = \quad (11)$$

$$= -e^x [x(y' - y'') - (y - y'')] = -e^x [(1-x)y'' + xy' - y] = 0 \quad (12)$$

$$(1-x)y'' + xy' - y = 0$$

$$679) y_1 = x, y_2 = x^2, y_3 = e^x$$

$$\begin{vmatrix} x & x^2 & e^x & y \\ 1 & 2x & e^x & y' \\ 0 & 2 & e^x & y'' \\ 0 & 0 & e^x & y''' \end{vmatrix} = \begin{vmatrix} x & x^2 & 1 & y \\ 1 & 2x & 1 & y' \\ 0 & 2 & 1 & y'' \\ 0 & 0 & 1 & y''' \end{vmatrix} e^x = \begin{vmatrix} x & x^2 & 0 & y-y''' \\ 1 & 2x & 0 & y'-y''' \\ 0 & 2 & 0 & y''-y''' \\ 0 & 0 & 1 & y''' \end{vmatrix} e^x = -e^x \begin{vmatrix} x & x^2 & y-y''' \\ 1 & 2x & y'-y''' \\ 0 & 2 & y''-y''' \end{vmatrix} \quad \text{?} =$$

$$= -e^x \begin{vmatrix} 0 & -x^2 & y - y''' - x(y' - y''') \\ 1 & 2x & y' - y''' \\ 0 & 2 & y'' - y''' \end{vmatrix} = e^x \begin{vmatrix} -x^2 & y - y''' - x(y' - y''') \\ 2 & y' - y''' \\ y'' - y''' \end{vmatrix} = \quad (13)$$

$$= e^x \{ -x^2 (y'' - y''') - 2[y - y''' - x(y' - y''')] \} = e^x \{ (x^2 - 2x + 2) y''' - x^2 y'' + 2xy' - 2y \} = 0$$

$$(x^2 - 2x + 2) y''' - x^2 y'' + 2xy' - 2y = 0 \quad (14)$$

(674, 676)

3 Получение второго частного решения по первому (метод Остроградского-Лиувилля)

$$y'' + p(x) y' + q(x) y = 0 \quad (15)$$

$$y = C_1 y_1 + C_2 y_2 \quad (16)$$

y_1 — известно.

$$\begin{vmatrix} y_1 & y_2 & y \\ y_1' & y_2' & y' \\ y_1'' & y_2'' & y'' \end{vmatrix} = 0 \quad (17)$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} y'' - \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} y' + \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} y = 0 \quad (18)$$

$$y'' - \frac{\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} y' + \frac{\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} y = 0 \quad (19)$$

$$p = - \frac{\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad (20)$$

$$\frac{d}{dx} \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \frac{d}{dx} (y_1 y_2' - y_1' y_2) = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1 y_2'} = y_1 y_2'' - y_1'' y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} \quad (21)$$

$$p = - \frac{\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = - \frac{d}{dx} \ln \left\| \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right\| \quad (22)$$

$$\ln \left\| \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right\| = -P(x) + \tilde{A}_1 \quad (23)$$

$$\left\| \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right\| = e^{-P(x) + \tilde{A}_1} \quad (24)$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \pm e^{\tilde{A}_1} e^{-P(x)} \equiv A_1 e^{-P(x)} \quad (25)$$

$$y_1 y_2' - y_1' y_2 = A_1 e^{-P(x)} \quad (26)$$

$$y_2 = y_1 z \quad y_2' = y_1 z' + y_1' z \quad (27)$$

$$y_1 (y_1 z' + y_1' z) - y_1' y_1 z = A_1 e^{-P(x)} \quad (28)$$

$$y_1^2 z' + \cancel{y_1 y_1' z} - \cancel{y_1' y_1 z} = A_1 e^{-P(x)} \quad (29)$$

$$z' = A_1 e^{-P(x)} y_1^{-2} \quad (30)$$

$$z = A_1 Y + A_2 \quad (31)$$

$$y_2 = y_1 (A_1 Y + A_2) = A_1 Y y_1 + A_2 y_1 = A_1 Y y_1 + A_2 y_1 \quad (32)$$

При

$$A_1 = 1; \quad A_2 = 0 \quad \implies \quad y_2 = Y y_1. \quad (33)$$

682)

$$x^2 (x+1) y'' - 2y = 0, \quad y_1 = 1 + \frac{1}{x} \quad (34)$$

$$y'' - \frac{2}{x^2 (x+1)} y = 0 \quad (35)$$

$$p(x) = -\frac{d}{dx} \ln \left\| \begin{array}{cc} y_1 & y_2 \\ y'_1 & y'_2 \end{array} \right\| = 0 \quad (36)$$

$$\ln \left\| \begin{array}{cc} y_1 & y_2 \\ y'_1 & y'_2 \end{array} \right\| = \tilde{A}_1, \quad \left| \begin{array}{cc} y_1 & y_2 \\ y'_1 & y'_2 \end{array} \right| = A_1, \quad y_1 y'_2 - y'_1 y_2 = A_1 \quad (37)$$

$$y_2 = y_1 z \quad y'_2 = y_1 z' + y'_1 z \quad (38)$$

$$y_1^2 z' = A_1 \quad (39)$$

$$z' = A_1 y_1^{-2} = A_1 \frac{1}{\left(1 + \frac{1}{x}\right)^2} = A_1 \frac{x^2}{(x+1)^2} \quad (40)$$

$$z = A_1 \left[x - \frac{1}{1+x} - 2 \ln |1+x| \right] + A_2 \quad (41)$$

$$y_2 = z y_1 \Big|_{\substack{A_1=1 \\ A_2=0}} = \left[x - \frac{1}{1+x} - 2 \ln |1+x| \right] \left(1 + \frac{1}{x} \right) \quad (42)$$

$$y = \left\{ C_1 + C_2 \left[x - \frac{1}{1+x} - 2 \ln |1+x| \right] \right\} \left(1 + \frac{1}{x} \right) \quad (43)$$

(681, 684)