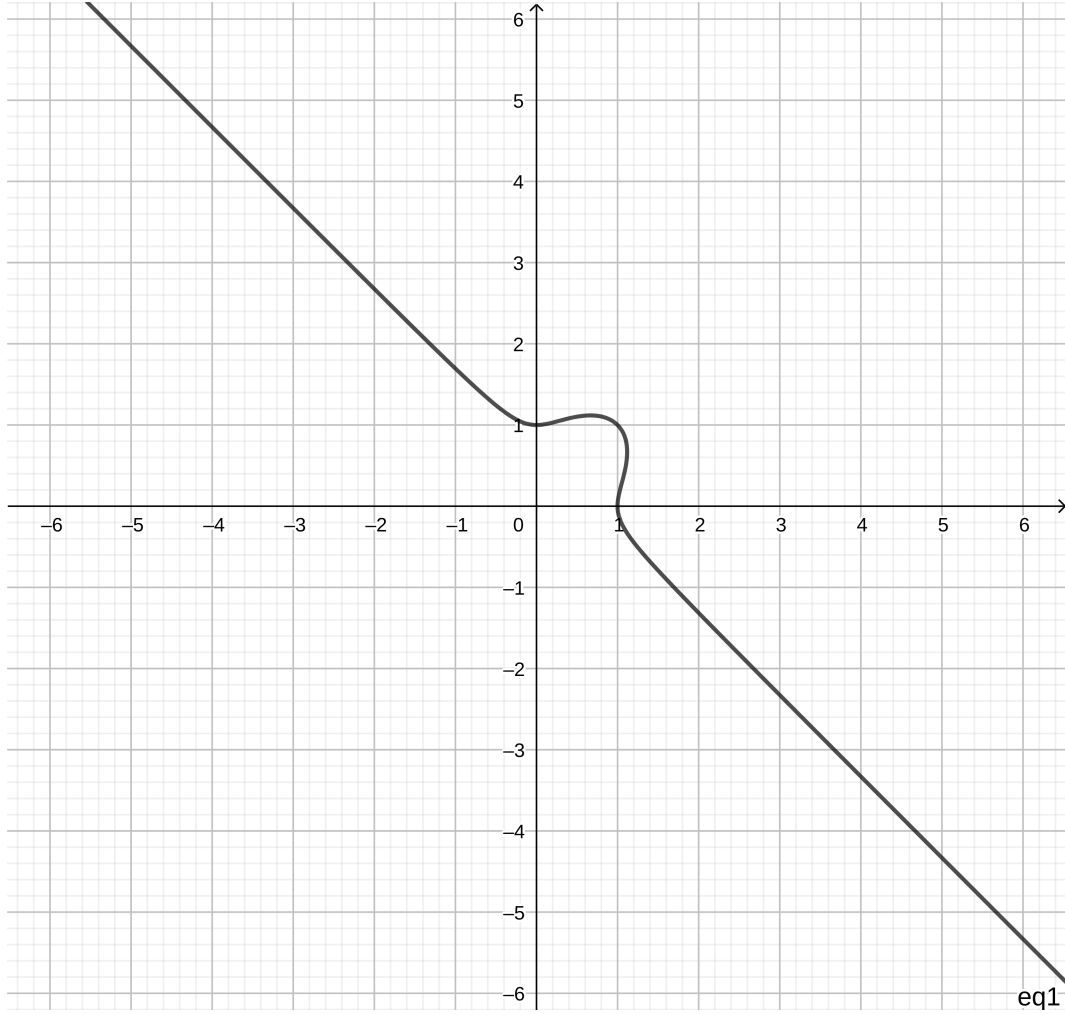


Вводя обобщённые полярные координаты, найти площадь, ограниченную кривыми:

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} = \frac{x^2}{h^2} + \frac{y^2}{k^2}, \quad x = 0, \quad y = 0. \quad (1)$$

чертёж при $a = b = h = k = 1$:



$x \geq 0, y \geq 0$
координаты:

$$\begin{cases} x = ar \cos^{2/3}(\varphi), \\ y = br \sin^{2/3}(\varphi). \end{cases} \quad (2)$$

Якобиан:

$$\begin{aligned} J &= \begin{vmatrix} x'_r & x'_\varphi \\ y'_r & y'_\varphi \end{vmatrix} = \begin{vmatrix} a \cos^{2/3}(\varphi) & -\frac{2}{3}ar \cos^{-1/3}(\varphi) \sin(\varphi) \\ b \sin^{2/3}(\varphi) & \frac{2}{3}br \sin^{-1/3}(\varphi) \cos(\varphi) \end{vmatrix} = \frac{2}{3}abr \begin{vmatrix} \cos^{2/3}(\varphi) & -\cos^{-1/3}(\varphi) \sin(\varphi) \\ \sin^{2/3}(\varphi) & \sin^{-1/3}(\varphi) \cos(\varphi) \end{vmatrix} = \\ &= \frac{2}{3}abr \left\{ \cos^{2/3}(\varphi) \sin^{-1/3}(\varphi) \cos(\varphi) + \sin^{2/3}(\varphi) \cos^{-1/3}(\varphi) \sin(\varphi) \right\} = \frac{2}{3}abr \left\{ \cos^{5/3}(\varphi) \sin^{-1/3}(\varphi) + \sin^{5/3}(\varphi) \cos^{-1/3}(\varphi) \right\} = \\ &= \frac{2}{3}abr \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) \left\{ \cos^{6/3}(\varphi) + \sin^{6/3}(\varphi) \right\} = \frac{2}{3}abr \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi). \end{aligned} \quad (3)$$

Уравнение кривой (R - значение r на границе фигуры):

$$\frac{(aR \cos^{2/3}(\varphi))^3}{a^3} + \frac{(bR \sin^{2/3}(\varphi))^3}{b^3} = \frac{(aR \cos^{2/3}(\varphi))^2}{h^2} + \frac{(bR \sin^{2/3}(\varphi))^2}{k^2}, \quad (4)$$

$$R^3 \cos^2(\varphi) + R^3 \sin^2(\varphi) = \frac{a^2 R^2 \cos^{4/3}(\varphi)}{h^2} + \frac{b^2 R^2 \sin^{4/3}(\varphi)}{k^2}, \quad (5)$$

$$R^3 = R^2 \left(\frac{a^2}{h^2} \cos^{4/3}(\varphi) + \frac{b^2}{k^2} \sin^{4/3}(\varphi) \right), \quad (6)$$

$$R = \frac{a^2}{h^2} \cos^{4/3}(\varphi) + \frac{b^2}{k^2} \sin^{4/3}(\varphi). \quad (7)$$

Интеграл площади:

$$S = \iint_{\Omega} 1 \cdot dx dy = \int_0^{\pi/2} d\varphi \int_0^R |J| dr, \quad (8)$$

по r :

$$\begin{aligned} \int_0^R |J| dr &= \int_0^R \frac{2}{3} ab r \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) dr = \frac{2}{3} ab \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) \int_0^R r dr = \\ &= \frac{2}{3} ab \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) \frac{r^2}{2} \Big|_0^R = \frac{1}{3} ab R^2 \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) = \\ &= \frac{1}{3} ab \left(\frac{a^2}{h^2} \cos^{4/3}(\varphi) + \frac{b^2}{k^2} \sin^{4/3}(\varphi) \right)^2 \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) = \\ &= \frac{1}{3} ab \left(\frac{a^4}{h^4} \cos^{8/3}(\varphi) + 2 \frac{a^2}{h^2} \frac{b^2}{k^2} \cos^{4/3}(\varphi) \sin^{4/3}(\varphi) + \frac{b^4}{k^4} \sin^{8/3}(\varphi) \right) \sin^{-1/3}(\varphi) \cos^{-1/3}(\varphi) = \\ &= \frac{1}{3} ab \left(\frac{a^4}{h^4} \cos^{7/3}(\varphi) \sin^{-1/3}(\varphi) + 2 \frac{a^2}{h^2} \frac{b^2}{k^2} \cos(\varphi) \sin(\varphi) + \frac{b^4}{k^4} \sin^{7/3}(\varphi) \cos^{-1/3}(\varphi) \right). \end{aligned} \quad (9)$$

Теперь по φ :

$$\begin{aligned} S &= \int_0^{\pi/2} d\varphi \int_0^R |J| dr = \\ &= \frac{ab}{3} \int_0^{\pi/2} \left(\frac{a^4}{h^4} \cos^{7/3}(\varphi) \sin^{-1/3}(\varphi) + 2 \frac{a^2}{h^2} \frac{b^2}{k^2} \cos(\varphi) \sin(\varphi) + \frac{b^4}{k^4} \sin^{7/3}(\varphi) \cos^{-1/3}(\varphi) \right) d\varphi = \\ &= \frac{ab}{3} \left(\frac{a^4}{h^4} \int_0^{\pi/2} \cos^{7/3}(\varphi) \sin^{-1/3}(\varphi) d\varphi + 2 \frac{a^2}{h^2} \frac{b^2}{k^2} \int_0^{\pi/2} \cos(\varphi) \sin(\varphi) d\varphi + \frac{b^4}{k^4} \int_0^{\pi/2} \sin^{7/3}(\varphi) \cos^{-1/3}(\varphi) d\varphi \right). \end{aligned} \quad (10)$$

Первое слагаемое (10):

$$\int_0^{\pi/2} \cos^{7/3} \varphi \sin^{-1/3} \varphi d\varphi = \int_0^{\pi/2} \cos^{4/3} \varphi \sin^{-4/3} \varphi \cos \varphi \sin \varphi d\varphi = \int_0^{\pi/2} (\cos^2 \varphi)^{2/3} (\sin^2 \varphi)^{-2/3} \cos \varphi \sin \varphi d\varphi =$$

$\sin^2 \varphi = t$

$$= \frac{1}{2} \int_0^{\pi/2} (1 - \sin^2 \varphi)^{2/3} (\sin^2 \varphi)^{-2/3} (\sin^2 \varphi)' d\varphi = \frac{1}{2} \int_0^1 (1 - t)^{2/3} t^{-2/3} dt \quad (11)$$

Подстановка Чебышова: $p = -\frac{2}{3}$, $q = 1$, $r = \frac{2}{3}$

$$\frac{p+1}{q} = \frac{-\frac{2}{3}+1}{1} = -\frac{2}{3}+1 = \frac{1}{3} \notin \mathbb{Z} \quad (12)$$

$$\frac{p+1}{q} + r = -\frac{2}{3} + 1 + \frac{2}{3} = 1 \in \mathbb{Z} \quad (13)$$

Третий случай: $\frac{1}{t} - 1 = z^3$, $t = \frac{1}{z^3+1}$

$$\begin{aligned} \int_0^1 (1-t)^{2/3} t^{-2/3} dt &= \int_0^1 \left(\frac{1}{t} - 1 \right)^{2/3} dt = \int_{-\infty}^0 (z^3)^{2/3} d \frac{1}{z^3+1} = - \int_0^{\infty} z^2 d \frac{1}{z^3+1} = \\ &= - \frac{z^2}{z^3+1} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{z^3+1} dz^2 = 2 \int_0^{\infty} \frac{z}{z^3+1} dz. \end{aligned} \quad (14)$$

Из (11) и (14):

$$\int_0^{\pi/2} \cos^{7/3} \varphi \sin^{-1/3} \varphi d\varphi = \frac{1}{2} \int_0^1 (1-t)^{2/3} t^{-2/3} dt = \int_0^\infty \frac{z}{z^3+1} dz. \quad (15)$$

$$\frac{z}{z^3+1} = \frac{z}{(z+1)(z^2-z+1)} = \frac{A}{z+1} + \frac{Bz+C}{z^2-z+1} = \frac{A(z^2-z+1) + (Bz+C)(z+1)}{(z+1)(z^2-z+1)} \quad (16)$$

$$z = A(z^2 - z + 1) + (Bz + C)(z + 1) = Az^2 - Az + A + Bz^2 + Bz + Cz + C \quad (17)$$

$$\begin{cases} A + B = 0 \\ -A + B + C = 1 \\ A + C = 0 \end{cases} \quad (18)$$

$$B = C = -A, \quad -A - A - A = 1 \quad (19)$$

$$A = -\frac{1}{3}, \quad B = C = \frac{1}{3} \quad (20)$$

$$\frac{z}{z^3+1} = -\frac{1}{3} \frac{1}{z+1} + \frac{1}{3} \frac{z+1}{z^2-z+1} \quad (21)$$

$$\int \frac{z}{z^3+1} dz = \int \left\{ -\frac{1}{3} \frac{1}{z+1} + \frac{1}{3} \frac{z+1}{z^2-z+1} \right\} dz = -\frac{1}{3} \ln|z+1| + \frac{1}{3} \int \frac{z+1}{z^2-z+1} dz, \quad (22)$$

$$z^2 - z + 1 = \left(z^2 - 2z \frac{1}{2} + \frac{1}{4} \right) - \frac{1}{4} + 1 = \left(z - \frac{1}{2} \right)^2 + \frac{3}{4} = \frac{3}{4} \left[\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1 \right]. \quad (23)$$

$$\frac{2z-1}{\sqrt{3}} = t, \quad z = \frac{\sqrt{3}t+1}{2}, \quad dz = \frac{\sqrt{3}}{2} dt$$

$$\begin{aligned} \int \frac{z+1}{z^2-z+1} dz &= \int \frac{z+1}{\frac{3}{4} \left[\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1 \right]} dz = \frac{4}{3} \int \frac{\frac{\sqrt{3}t+1}{2} + 1}{t^2+1} \frac{\sqrt{3}}{2} dt = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}t+3}{t^2+1} dt = \\ &= \frac{1}{\sqrt{3}} \left(\sqrt{3} \int \frac{t}{t^2+1} dt + 3 \int \frac{dt}{t^2+1} \right) = \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}}{2} \ln(t^2+1) + 3 \operatorname{arctg} t \right) + C = \frac{1}{2} \ln(t^2+1) + \sqrt{3} \operatorname{arctg} t + C = \\ &= \frac{1}{2} \ln \left[\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1 \right] + \sqrt{3} \operatorname{arctg} \frac{2z-1}{\sqrt{3}} + C, \end{aligned} \quad (24)$$

$$\begin{aligned} \int \frac{z}{z^3+1} dz &= -\frac{1}{3} \ln|z+1| + \frac{1}{3} \int \frac{z+1}{z^2-z+1} dz = -\frac{1}{3} \ln|z+1| + \frac{1}{3} \left\{ \frac{1}{2} \ln \left[\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1 \right] + \sqrt{3} \operatorname{arctg} \frac{2z-1}{\sqrt{3}} + C \right\} = \\ &= \frac{1}{6} \left\{ \ln \left[\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1 \right] - \ln|z+1|^2 \right\} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2z-1}{\sqrt{3}} + \frac{C}{3} = \\ &= \frac{1}{6} \ln \left[\frac{\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1}{|z+1|^2} \right] + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2z-1}{\sqrt{3}} + \frac{C}{3}, \end{aligned} \quad (25)$$

$$\begin{aligned} \int_0^\infty \frac{z}{z^3+1} dz &= \left\{ \frac{1}{6} \ln \left[\frac{\left(\frac{2z-1}{\sqrt{3}} \right)^2 + 1}{|z+1|^2} \right] + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2z-1}{\sqrt{3}} \right\} \bigg|_0^\infty = \frac{1}{6} \left(\ln \frac{4}{3} - \ln \frac{4}{3} \right) + \frac{1}{\sqrt{3}} \left[\frac{\pi}{2} - \operatorname{arctg} \left(-\frac{1}{\sqrt{3}} \right) \right] = \\ &= \frac{1}{\sqrt{3}} \left[\frac{\pi}{2} + \frac{\pi}{6} \right] = \frac{1}{\sqrt{3}} \frac{4\pi}{6} = \frac{2\pi}{3\sqrt{3}}. \end{aligned} \quad (26)$$

Из (15)

$$\int_0^{\pi/2} \cos^{7/3} \varphi \sin^{-1/3} \varphi d\varphi = \int_0^\infty \frac{z}{z^3+1} dz = \frac{2\pi}{3\sqrt{3}} \quad (27)$$

Третье слагаемое (10) ($\varphi = \frac{\pi}{2} - \psi$):

$$\int_0^{\pi/2} \sin^{7/3} \varphi \cos^{-1/3} \varphi d\varphi = - \int_{\pi/2}^0 \sin^{7/3} \left(\frac{\pi}{2} - \psi \right) \cos^{-1/3} \left(\frac{\pi}{2} - \psi \right) d\psi = \int_0^{\pi/2} \cos^{7/3} \psi \sin^{-1/3} \psi d\psi = \frac{2\pi}{3\sqrt{3}} \quad (28)$$

Второе слагаемое (10)

$$\int_0^{\pi/2} \cos(\varphi) \sin(\varphi) d\varphi = \frac{1}{2} \int_0^{\pi/2} \sin(2\varphi) d\varphi = -\frac{1}{4} \cos(2\varphi) \Big|_0^{\pi/2} = -\frac{1}{4} (-1 - 1) = \frac{1}{2}. \quad (29)$$

Тогда, окончательно, сама площадь равна

$$S = \frac{ab}{3} \left(\frac{a^4}{h^4} \frac{2\pi}{3\sqrt{3}} + 2 \frac{a^2}{h^2} \frac{b^2}{k^2} \frac{1}{2} + \frac{b^4}{k^4} \frac{2\pi}{3\sqrt{3}} \right) = \frac{ab}{3} \left[\frac{2\pi}{3\sqrt{3}} \left(\frac{a^4}{h^4} + \frac{b^4}{k^4} \right) + \frac{a^2}{h^2} \frac{b^2}{k^2} \right]. \quad (30)$$