

$$V = \iint_{\Omega} f(x, y) \, dx dy \quad (1)$$

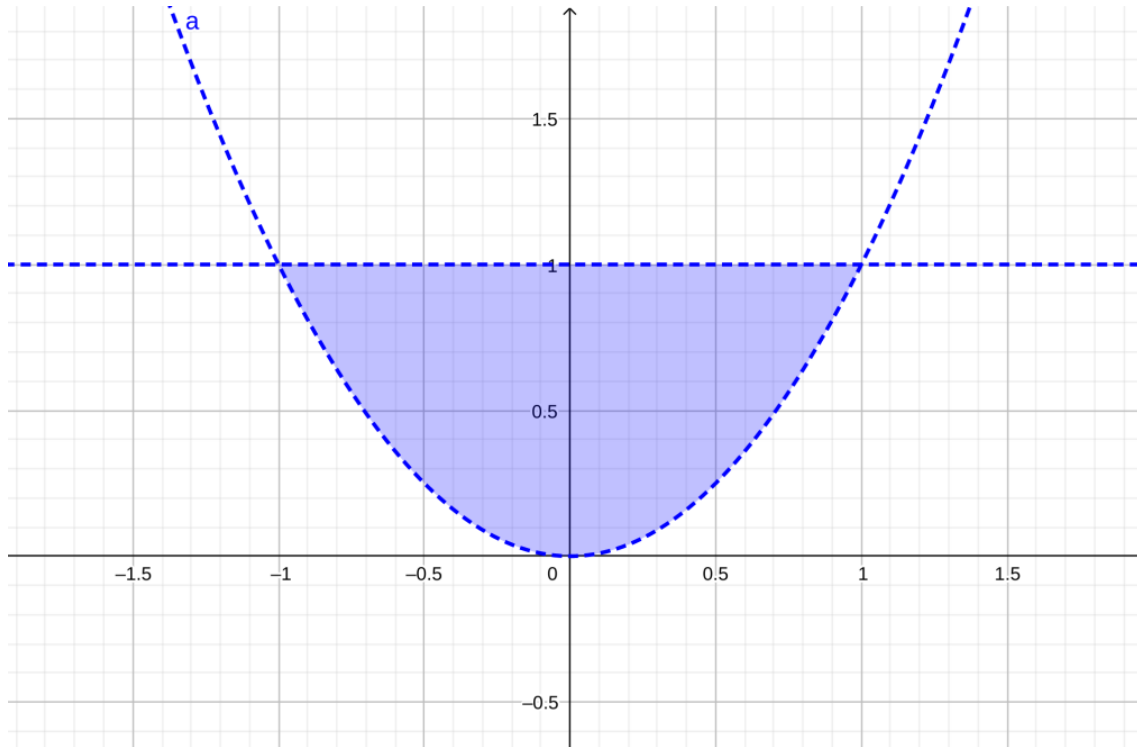
$$V = \iint_{\Omega} (z_{sup} - z_{inf}) \, dx dy \quad (2)$$

1 Демидович № 4009

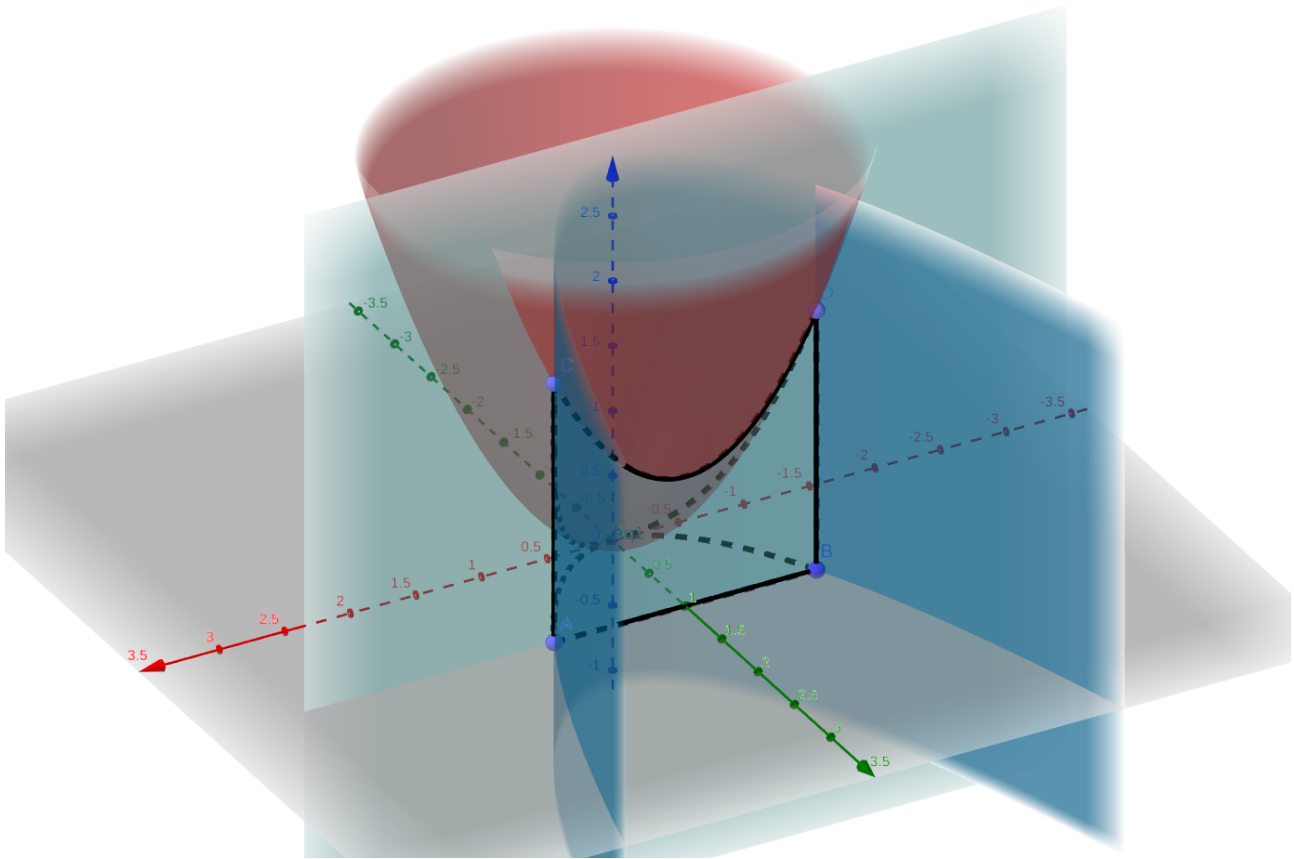
Найти объём тела, ограниченного поверхностями:

$$z = x^2 + y^2, \quad y = x^2, \quad y = 1, \quad z = 0. \quad (3)$$

$$y = x^2, \quad y = 1 : \quad (4)$$



В объёме:



$$\begin{aligned}
 V &= \iint_S z(x, y) \, dxdy = \iint_S (x^2 + y^2) \, dxdy = \int_{-1}^1 dx \int_{x^2}^1 dy (x^2 + y^2) = \int_{-1}^1 dx \left(x^2 y + \frac{y^3}{3} \right) \Big|_{x^2}^1 = \\
 &= \int_{-1}^1 dx \left[\left(x^2 + \frac{1}{3} \right) - \left(x^2 x^2 + \frac{(x^2)^3}{3} \right) \right] = \int_{-1}^1 \left[\frac{1}{3} + x^2 - x^4 - \frac{x^6}{3} \right] dx = \left[\frac{1}{3}x + \frac{x^3}{3} - \frac{x^5}{5} - \frac{x^7}{21} \right] \Big|_{-1}^1 = \\
 &= 2 \left[\frac{1}{3} + \frac{1}{3} - \frac{1}{5} - \frac{1}{21} \right] = 2 \left[\frac{13}{21} - \frac{1}{5} \right] = 2 \left[\frac{65}{21 \cdot 5} - \frac{21}{21 \cdot 5} \right] = \frac{88}{105}.
 \end{aligned} \tag{5}$$

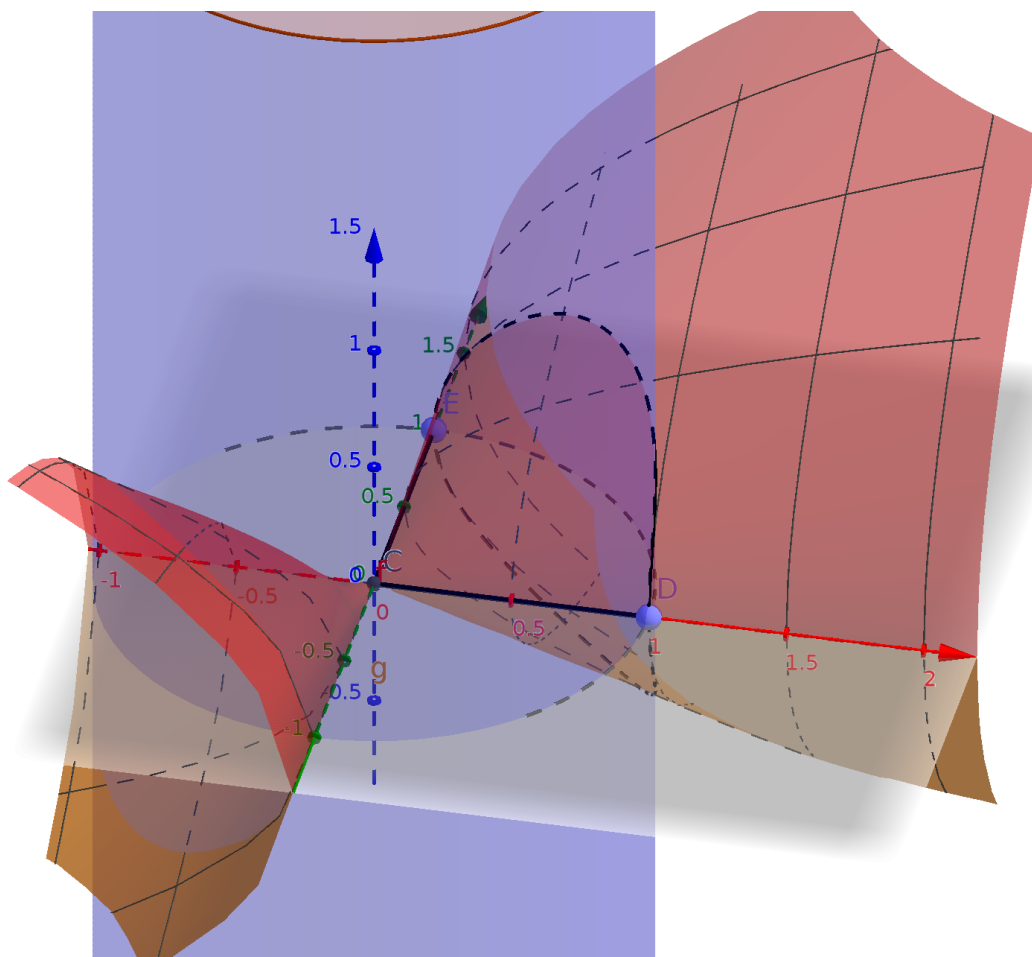
Задание: № 4007

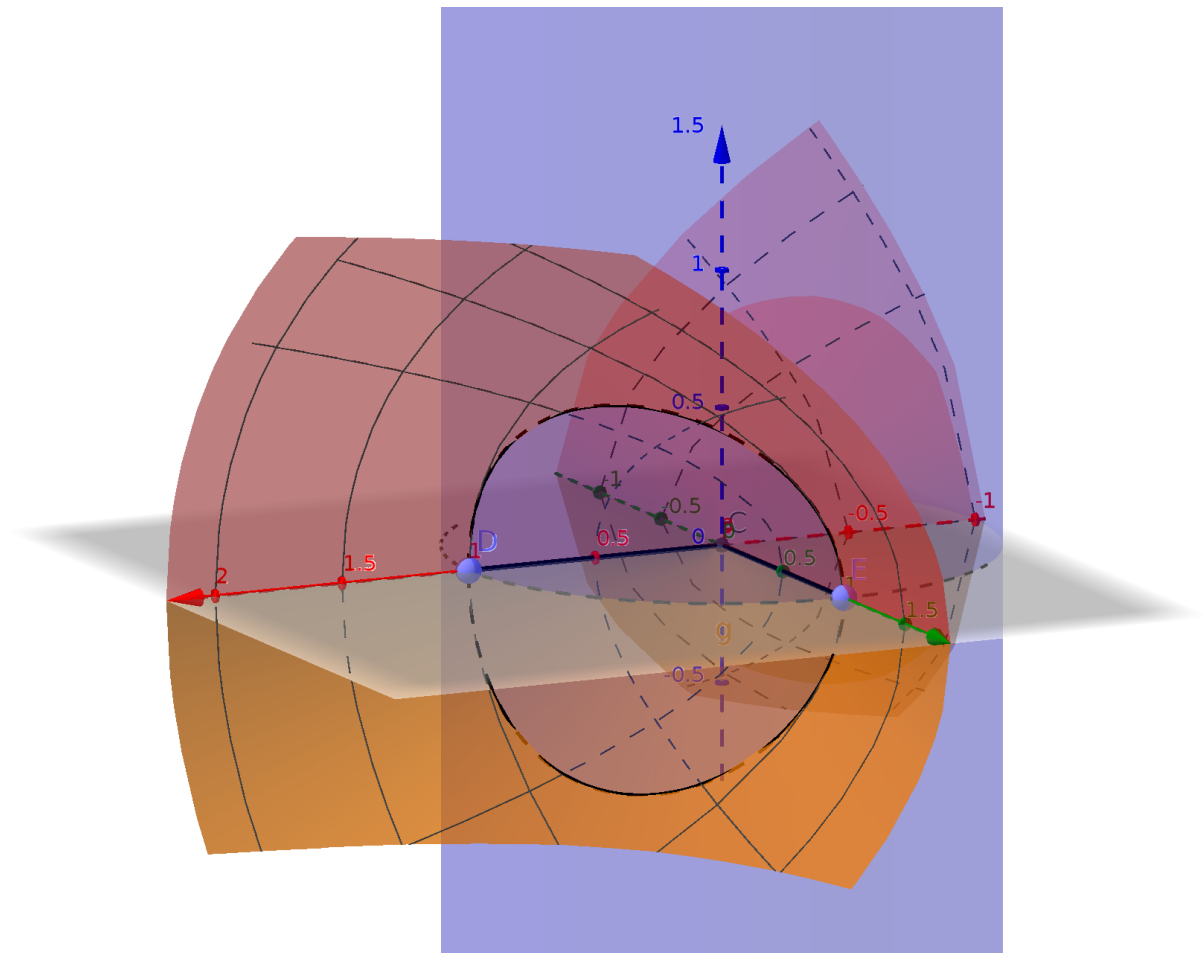
2 Демидович № 4013

Переходя к полярным координатам, найти объемы тел, ограниченных следующими поверхностями:

$$z^2 = xy, \quad x^2 + y^2 = a^2 \quad (6)$$

$$z_1 = \sqrt{xy}, \quad z_2 = -\sqrt{xy}; \quad xy \geq 0 \quad (7)$$





$$V = \iint_{\Omega} (z_1 - z_2) dx dy = \iint_{\Omega} 2\sqrt{xy} dx dy \quad (8)$$

$$\begin{cases} x = r \cos \varphi, \\ y = r \sin \varphi; \end{cases} \quad (9)$$

$$|J| = r. \quad (10)$$

$$V = \int_0^{\pi/2} d\varphi \int_0^a dr \sqrt{r \cos \varphi r \sin \varphi} = \int_0^{\pi/2} d\varphi \sqrt{\cos \varphi \sin \varphi} \int_0^a dr r^2 = \frac{a^3}{3} \int_0^{\pi/2} \sqrt{\cos \varphi \sin \varphi} d\varphi. \quad (11)$$

$$\sin^2 \varphi = t$$

$$\begin{aligned} V &= \frac{a^3}{3} \int_0^{\pi/2} \sqrt{\cos \varphi \sin \varphi} d\varphi = \frac{a^3}{3} \int_0^{\pi/2} \cos^{-1/2} \varphi \sin^{-1/2} \varphi \cos \varphi \sin \varphi d\varphi = \frac{a^3}{6} \int_0^{\pi/2} (\sin^2 \varphi)^{-1/4} (\cos^2 \varphi)^{-1/4} (\sin^2 \varphi)' d\varphi = \\ &= \frac{a^3}{6} \int_0^1 t^{-1/4} (1-t)^{-1/4} dt = \frac{a^3}{6} \int_0^1 t^{3/4-1} (1-t)^{3/4-1} dt = \frac{a^3}{6} B\left(\frac{3}{4}, \frac{3}{4}\right) = \end{aligned} \quad (12)$$

здесь $B\left(\frac{3}{4}, \frac{3}{4}\right)$ — это интеграл Эйлера 1-го рода

$$= \frac{a^3}{6} \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{6}{4}\right)} = \frac{a^3 \Gamma^2\left(\frac{3}{4}\right)}{3\sqrt{\pi}}.$$

Задание: № 4014

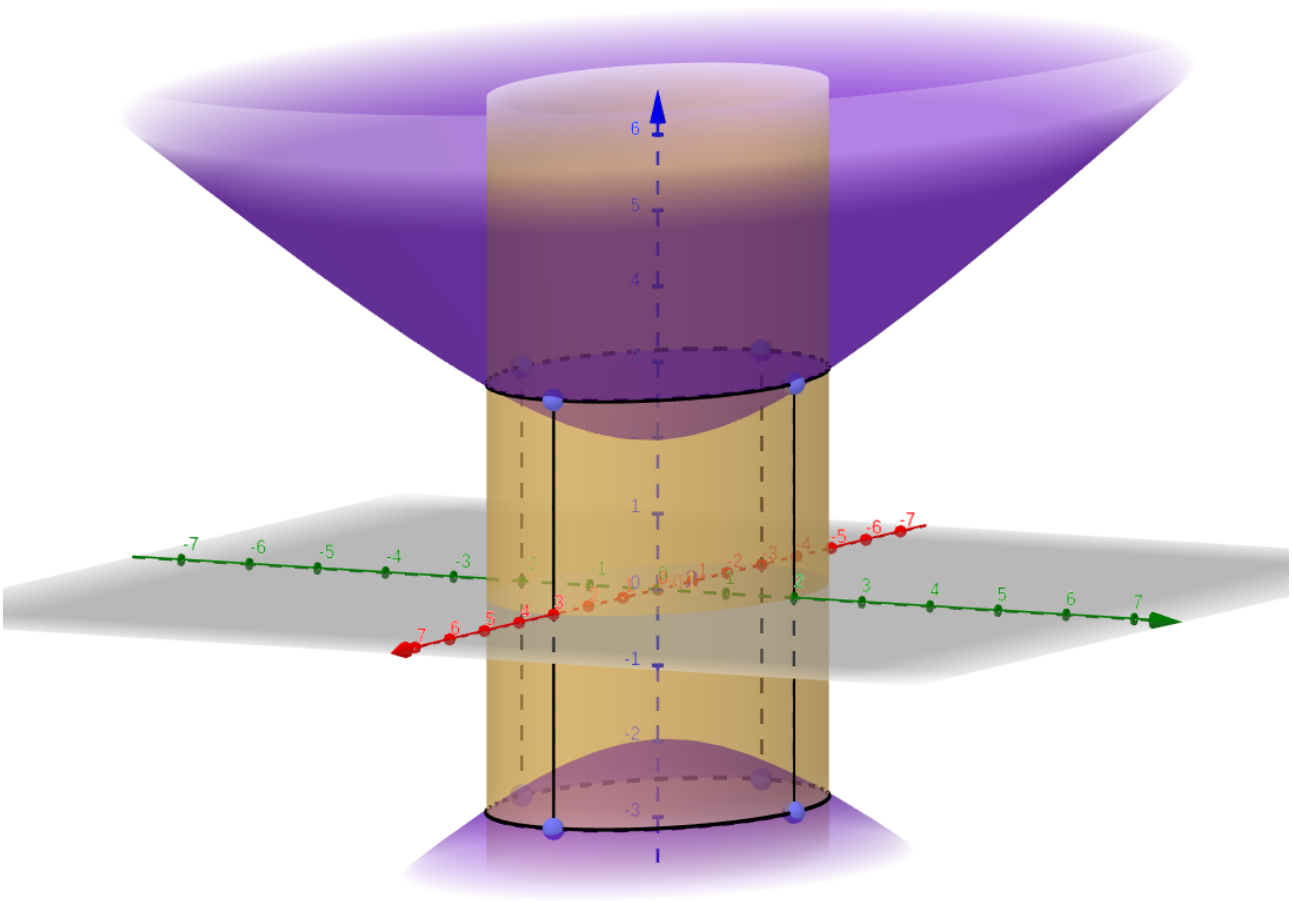
3 Демидович № 4022

Найти объём тела, ограниченного поверхностями ($a, b, c > 0$):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (13)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = \frac{z^2}{c^2}, \quad (14)$$

$$z = \pm c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}. \quad (15)$$



$$V = \iint_S (z_{sup} - z_{inf}) dxdy = \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} 2c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1} dxdy. \quad (16)$$

$$\begin{cases} x = ar \cos \varphi, \\ y = br \sin \varphi. \end{cases} \quad (17)$$

$$J = abr. \quad (18)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2 r^2 \cos^2 \varphi}{a^2} + \frac{b^2 r^2 \sin^2 \varphi}{b^2} = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi = r^2. \quad (19)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \implies r^2 \leq 1, r \leq 1.$$

$$V = \iint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} 2c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1} dxdy = 2c \int_0^{2\pi} d\varphi \int_0^1 dr \sqrt{r^2 + 1} \cdot abr = 2abc \cdot 2\pi \cdot \frac{1}{3} (r^2 + 1)^{\frac{3}{2}} \Big|_0^1 = \frac{4\pi}{3} [2\sqrt{2} - 1] abc. \quad (20)$$

Задание: № 4021