1 Условия задачи

$$\Delta U = 0 \tag{1}$$

$$U|_{\rho=l} = \begin{cases} U_0, & 0 \leqslant \theta \leqslant \frac{\pi}{2}, \\ -U_0, & \frac{\pi}{2} \leqslant \theta \leqslant \pi. \end{cases} \tag{2}$$

2 Переход в сферические координаты

Переведём лапласиан в сферическую систему координат. В декартовой и циллиндрической системе

$$\Delta U = U_{xx} + U_{yy} + U_{zz} = U_{rr} + \frac{1}{r^2}U_{\varphi\varphi} + \frac{1}{r}U_r + U_{zz} = \frac{1}{r}\frac{\partial}{\partial r}(rU_r) + \frac{1}{r^2}U_{\varphi\varphi} + U_{zz}. \tag{3}$$

Переход между циллиндрической и сферической системами координат

$$\begin{cases} r = \rho\sin\theta \\ \varphi = \varphi \\ z = \rho\cos\theta \end{cases} \tag{4}$$

продифференцируем по каждой из циллиндрических координат:

$$\begin{cases} 1 = \rho'_r\sin\theta + \rho\cos\theta\theta'_r \\ 0 = \varphi'_r \\ 0 = \rho'_r\cos\theta - \rho\sin\theta\theta'_r \end{cases} \qquad \begin{cases} 0 = \rho'_\varphi\sin\theta + \rho\cos\theta\theta'_\varphi \\ 1 = \varphi'_\varphi \\ 0 = \rho'_\varphi\cos\theta - \rho\sin\theta\theta'_\varphi \end{cases} \qquad \begin{cases} 0 = \rho'_z\sin\theta + \rho\cos\theta\theta'_z \\ 0 = \varphi'_z \\ 1 = \rho'_z\cos\theta - \rho\sin\theta\theta'_z \end{cases} \tag{5}$$

Отсюда получим

$$\varphi'_r = \varphi'_z = 0, \qquad \rho'_\varphi = \theta'_\varphi = 0; \tag{6}$$

$$\rho'_r = \sin\theta, \quad \theta'_r = \frac{\cos\theta}{\rho}; \tag{7}$$

$$\rho'_z = \cos\theta, \quad \theta'_z = -\frac{\sin\theta}{\rho}. \tag{8}$$

Преобразуем по одному слагаемые ΔU в циллиндрической системе. Начнём с $\frac{1}{r}\frac{\partial}{\partial r}(rU_r)$.

$$\frac{\partial}{\partial r} = \frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial r}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial r}\frac{\partial}{\partial\varphi} = \sin\theta\frac{\partial}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial}{\partial\theta}, \tag{9}$$

$$U_r = \sin\theta U_\rho + \frac{\cos\theta}{\rho}U_\theta, \tag{10}$$

$$rU_r = \rho\sin^2\theta U_\rho + \frac{1}{2}\sin(2\theta)U_\theta, \tag{11}$$

$$\frac{\partial}{\partial r}(rU_r) = \left(\sin\theta\frac{\partial}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial}{\partial\theta}\right)\left(\rho\sin^2\theta U_\rho + \frac{1}{2}\sin(2\theta)U_\theta\right) = \tag{12}$$

$$= \sin^3\theta\frac{\partial}{\partial\rho}\rho U_\rho + \cos\theta\frac{\partial}{\partial\theta}\sin^2\theta U_\rho + \frac{1}{2}\sin(2\theta)\sin\theta\frac{\partial}{\partial\rho}U_\theta + \frac{\cos\theta}{2\rho}\frac{\partial}{\partial\theta}\sin(2\theta)U_\theta,$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right) = \frac{1}{\rho}\sin^2\theta\frac{\partial}{\partial\rho}\rho U_\rho + \frac{\cos\theta}{\rho\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta U_\rho + \frac{1}{2\rho}\sin\left(2\theta\right)\frac{\partial}{\partial\rho}U_\theta + \frac{\cos\theta}{2\rho^2\sin\theta}\frac{\partial}{\partial\theta}\sin\left(2\theta\right)U_\theta =$$

$$= \frac{1}{\rho}\sin^2\theta\left(\rho U_{\rho\rho} + U_\rho\right) + \frac{\cos\theta}{\rho\sin\theta}\left(\sin\left(2\theta\right)U_\rho + \sin^2\theta U_{\rho\theta}\right) + \frac{1}{2\rho}\sin\left(2\theta\right)U_{\theta\rho} + \frac{\cos\theta}{2\rho^2\sin\theta}\left(2\cos\left(2\theta\right)U_\theta + \sin\left(2\theta\right)U_{\theta\theta}\right) = \quad (13)$$

$$= \left(\sin^2\theta U_{\rho\rho} + \frac{1}{\rho}\sin^2\theta U_\rho\right) + \left(\frac{2\cos^2\theta}{\rho}U_\rho + \frac{\sin 2\theta}{2\rho}U_{\rho\theta}\right) + \frac{\sin\left(2\theta\right)}{2\rho}U_{\theta\rho} + \left(\frac{\cos\theta\cos\left(2\theta\right)}{\rho^2\sin\theta}U_\theta + \frac{\cos^2\theta}{\rho^2}U_{\theta\theta}\right) =$$

$$= \sin^2\theta U_{\rho\rho} + \frac{\sin 2\theta}{\rho}U_{\rho\theta} + \frac{\cos^2\theta}{\rho^2}U_{\theta\theta} + \frac{1+\cos^2\theta}{\rho}U_\rho + \frac{\cos\theta\cos\left(2\theta\right)}{\rho^2\sin\theta}U_\theta.$$

Теперь U_{zz}:

$$\frac{\partial}{\partial z} = \frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi} = \cos\theta\frac{\partial}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}, \quad (14)$$

$$U_{zz} = \frac{\partial}{\partial z}U_z = \left(\cos\theta\frac{\partial}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right)\left(\cos\theta U_\rho - \frac{\sin\theta}{\rho}U_\theta\right) =$$

$$= \cos^2\theta\frac{\partial}{\partial\rho}U_\rho - \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\cos\theta U_\rho - \frac{1}{2}\sin\left(2\theta\right)\frac{\partial}{\partial\rho}\frac{1}{\rho}U_\theta + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\sin\theta U_\theta =$$

$$= \cos^2\theta U_{\rho\rho} - \frac{\sin\theta}{\rho}\left(-\sin\theta U_\rho + \cos\theta U_{\rho\theta}\right) - \frac{1}{2}\sin\left(2\theta\right)\left(-\frac{1}{\rho^2}U_\theta + \frac{1}{\rho}U_{\theta\rho}\right) + \frac{\sin\theta}{\rho^2}\left(\cos\theta U_\theta + \sin\theta U_{\theta\theta}\right) = \quad (15)$$

$$= \cos^2\theta U_{\rho\rho} + \frac{\sin^2\theta}{\rho}U_\rho - \frac{\sin\left(2\theta\right)}{2\rho}U_{\rho\theta} + \frac{1}{2\rho^2}\sin\left(2\theta\right)U_\theta - \frac{1}{2\rho}\sin\left(2\theta\right)U_{\theta\rho} + \frac{\sin\left(2\theta\right)}{2\rho^2}U_\theta + \frac{\sin^2\theta}{\rho^2}U_{\theta\theta} =$$

$$= \cos^2\theta U_{\rho\rho} - \frac{\sin\left(2\theta\right)}{\rho}U_{\rho\theta} + \frac{\sin^2\theta}{\rho^2}U_{\theta\theta} + \frac{\sin^2\theta}{\rho}U_\rho + \frac{\sin\left(2\theta\right)}{\rho^2}U_\theta.$$

Упростим сумму

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right) + U_{zz} = \sin^2\theta U_{\rho\rho} + \frac{\sin 2\theta}{\rho}U_{\rho\theta} + \frac{\cos^2\theta}{\rho^2}U_{\theta\theta} + \frac{1+\cos^2\theta}{\rho}U_\rho + \frac{\cos\theta\cos\left(2\theta\right)}{\rho^2\sin\theta}U_\theta +$$

$$+ \cos^2\theta U_{\rho\rho} - \frac{\sin\left(2\theta\right)}{\rho}U_{\rho\theta} + \frac{\sin^2\theta}{\rho^2}U_{\theta\theta} + \frac{\sin^2\theta}{\rho}U_\rho + \frac{\sin\left(2\theta\right)}{\rho^2}U_\theta =$$

$$= U_{\rho\rho} + \frac{1}{\rho^2}U_{\theta\theta} + \frac{2}{\rho}U_\rho + \left(\frac{\cos\left(2\theta\right)}{\sin\theta} + 2\sin\left(\theta\right)\right)\cos\theta\frac{U_\theta}{\rho^2} = \quad (16)$$

$$= U_{\rho\rho} + \frac{1}{\rho^2}U_{\theta\theta} + \frac{2}{\rho}U_\rho + \frac{\cos\theta}{\sin\theta}\frac{U_\theta}{\rho^2},$$

и, наконец, вычислим весь лапласиан:

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right) + \frac{1}{r^2}U_{\varphi\varphi} + U_{zz} = U_{\rho\rho} + \frac{1}{\rho^2}U_{\theta\theta} + \frac{1}{\rho^2\sin^2\theta}U_{\varphi\varphi} + \frac{2}{\rho}U_\rho + \frac{\cos\theta}{\sin\theta}\frac{U_\theta}{\rho^2} \quad (17)$$

3 Разделение переменных

Будем искать решения уравнения Лапласа, которое примет вид

$$U_{\rho\rho} + \frac{1}{\rho^2} U_{\theta\theta} + \frac{1}{\rho^2 \sin^2\theta} U_{\varphi\varphi} + \frac{2}{\rho} U_\rho + \frac{\cos\theta}{\sin\theta} \frac{U_\theta}{\rho^2} = 0, \tag{18}$$

в виде произведений

$$U(\rho, \theta, \varphi) = R(\rho)\,\Theta(\theta)\,\Phi(\varphi). \tag{19}$$

Как всегда,

$$\Phi(\varphi) = \Phi(\varphi + 2\pi). \tag{20}$$

Начнём

$$R''\Theta\Phi + \frac{1}{\rho^2} R\Theta''\Phi + \frac{1}{\rho^2 \sin^2\theta} R\Theta\Phi'' + \frac{2}{\rho} R'\Theta\Phi + \frac{\cos\theta}{\sin\theta} \frac{R\Theta'\Phi}{\rho^2} = 0 \Bigg| \cdot \frac{\rho^2 \sin^2\theta}{R\Theta\Phi} \tag{21}$$

$$\frac{R''}{R}\rho^2 \sin^2\theta + \frac{\Theta''}{\Theta}\sin^2\theta + \frac{\Phi''}{\Phi} + 2\sin^2\theta\rho\frac{R'}{R} + \sin\theta\cos\theta\frac{\Theta'}{\Theta} = 0 \tag{22}$$

$$\frac{R''}{R}\rho^2 \sin^2\theta + \frac{\Theta''}{\Theta}\sin^2\theta + 2\sin^2\theta\rho\frac{R'}{R} + \sin\theta\cos\theta\frac{\Theta'}{\Theta} = -\frac{\Phi''}{\Phi} = \mu, \quad \mu = const. \tag{23}$$

3.1 Азимутальная часть

$$-\frac{\Phi''}{\Phi} = \mu \tag{24}$$

из (24) и (20), как в задаче 60 получаем

$$\mu = n^2, \tag{25}$$

$$\Phi_n = C_n^1 \cos(n\varphi) + C_n^2 \sin(n\varphi). \tag{26}$$

$$\frac{R''}{R}\rho^2 \sin^2\theta + \frac{\Theta''}{\Theta}\sin^2\theta + 2\sin^2\theta\rho\frac{R'}{R} + \sin\theta\cos\theta\frac{\Theta'}{\Theta} = n^2 \Bigg| \cdot \frac{1}{\sin^2\theta} \tag{27}$$

$$\frac{R''}{R}\rho^2 + 2\rho\frac{R'}{R} + \frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} = \frac{n^2}{\sin^2\theta} \tag{28}$$

$$\frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} - \frac{n^2}{\sin^2\theta} = -\frac{R''}{R}\rho^2 - 2\rho\frac{R'}{R} = \varkappa, \quad \varkappa = const \tag{29}$$

3.2 Высотная часть

$$\frac{\Theta''}{\Theta} + \frac{\cos\theta}{\sin\theta}\frac{\Theta'}{\Theta} - \frac{n^2}{\sin^2\theta} = \varkappa \Bigg| \cdot \Theta \tag{30}$$

$$\Theta'' + \frac{\cos\theta}{\sin\theta}\Theta' + \left(-\varkappa - \frac{n^2}{\sin^2\theta}\right)\Theta = 0 \tag{31}$$

Произведём замену

$$\cos\theta = x \tag{32}$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2, \qquad \frac{dx}{d\theta} = -\sin\theta \tag{33}$$

$$\Theta' = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Theta}{dx}, \qquad \Theta'' = \frac{d}{d\theta}\Theta' = -\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{dx}\right) = \sin^2\theta\frac{d^2\Theta}{dx^2} - \cos\theta\frac{d\Theta}{dx} \tag{34}$$

$$\left(\sin^2\theta\frac{d^2\Theta}{dx^2} - \cos\theta\frac{d\Theta}{dx}\right) + \frac{\cos\theta}{\sin\theta}\left(-\sin\theta\frac{d\Theta}{dx}\right) + \left(-\varkappa - \frac{n^2}{\sin^2\theta}\right)\Theta = 0 \tag{35}$$

$$\sin^2\theta\frac{d^2\Theta}{dx^2} - \cos\theta\frac{d\Theta}{dx} - \cos\theta\frac{d\Theta}{dx} + \left(-\varkappa - \frac{n^2}{\sin^2\theta}\right)\Theta = 0 \tag{36}$$

$$\left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left(-\varkappa - \frac{n^2}{1 - x^2}\right)\Theta = 0 \tag{37}$$

Это уравнение имеет нетривиальные решения только при

$$-\varkappa = m\left(m+1\right), \tag{38}$$

которые записываются так:

$$\Theta = C^3_{nm}P^n_m\left(x\right) = C^3_{nm}P^n_m\left(\cos\theta\right). \tag{39}$$

3.3 Радиальная часть

$$-\frac{R''}{R}\rho^2 - 2\rho\frac{R'}{R} = \varkappa = -m\left(m+1\right), \tag{40}$$

$$R''\rho^2 + 2\rho R' - m\left(m+1\right)R = 0 \tag{41}$$

уравнение Эйлера; $\rho = e^x$ $x = \ln\rho$

$$R' = \frac{1}{\rho}\frac{dR}{dx}, \tag{42}$$

$$R'' = \frac{1}{\rho^2}\left(\frac{d^2R}{dx^2} - \frac{dR}{dx}\right), \tag{43}$$

$$\frac{d^2R}{dx^2} + \frac{dR}{dx} - m\left(m+1\right)R = 0 \tag{44}$$

ХУ:

$$\lambda^2 + \lambda - m\left(m+1\right) = 0 \tag{45}$$

$$\lambda = m, \quad \lambda = -m-1 \tag{46}$$

$$R = C^4_m e^{mx} + C^5_m e^{-(m+1)x} = C^4_m\rho^m + C^5_m\rho^{-(m+1)} = C^4_m\rho^m \tag{47}$$

Общее решение:

$$U = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C^4_m\rho^m C^3_{nm}P^n_m\left(\cos\theta\right)\left[C^1_n\cos\left(n\varphi\right) + C^2_n\sin\left(n\varphi\right)\right] =$$

$C^4_m C^3_{nm} C^1_n = C^6_{mn}$, $C^4_m C^3_{nm} C^2_n = C^7_{mn}$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho^m P^n_m\left(\cos\theta\right)\left[C^6_{mn}\cos\left(n\varphi\right) + C^7_{mn}\sin\left(n\varphi\right)\right] \tag{48}$$

4 Граничные условия

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} l^m P_m^n(\cos\theta)\left[C_{mn}^6\cos(n\varphi)+C_{mn}^7\sin(n\varphi)\right]=\begin{cases}U_0, & 0\leqslant\theta\leqslant\frac{\pi}{2},\\ -U_0, & \frac{\pi}{2}\leqslant\theta\leqslant\pi.\end{cases}\tag{49}$$

Т.к. граничные условия не зависят от φ, повторив выводы из задачи 114, получим, что

$$C_{mn}^7=0;\quad C_{mn}^6=0,\ n\neq 0.\tag{50}$$

Ненулевыми могут быть только C_{m0}^6, и

$$U=\sum_{m=0}^{\infty}C_{m0}^6\rho^m P_m(\cos\theta)\tag{51}$$

$$\sum_{m=0}^{\infty}C_{m0}^6 l^m P_m(\cos\theta)=\begin{cases}U_0, & 0\leqslant\theta\leqslant\frac{\pi}{2},\\ -U_0, & \frac{\pi}{2}\leqslant\theta\leqslant\pi.\end{cases}\tag{52}$$

Снова совершим замену (32):

$$\sum_{m=0}^{\infty}C_{m0}^6 l^m P_m(x)=f(x)=\begin{cases}U_0, & 0\leqslant x\leqslant 1,\\ -U_0, & -1\leqslant x\leqslant 0.\end{cases}\tag{53}$$

Чтобы найти C_{m0}^6, воспользуемся свойством

$$\int_{-1}^{1}P_m(x)P_k(x)\,dx=\delta_{mk}\frac{2}{2k+1}\tag{54}$$

$$\int_{-1}^{1}\sum_{m=0}^{\infty}C_{m0}^6 l^m P_m(x)P_k(x)\,dx=\int_{-1}^{1}f(x)P_k(x)\,dx\tag{55}$$

Левая часть:

$$\int_{-1}^{1}\sum_{m=0}^{\infty}C_{m0}^6 l^m P_m(x)P_k(x)\,dx=\sum_{m=0}^{\infty}C_{m0}^6 l^m\int_{-1}^{1}P_m(x)P_k(x)\,dx=\sum_{m=0}^{\infty}C_{m0}^6 l^m\delta_{mk}\frac{2}{2k+1}=C_{k0}^6\frac{2l^k}{2k+1}\tag{56}$$

Правая часть:

$$\int_{-1}^{1}f(x)P_k(x)\,dx=\int_{-1}^{0}f(x)P_k(x)\,dx+\int_{0}^{1}f(x)P_k(x)\,dx=-U_0\int_{-1}^{0}P_k(x)\,dx+U_0\int_{0}^{1}P_k(x)\,dx=$$

$$=U_0\left(\int_{0}^{1}P_k(x)\,dx-\int_{-1}^{0}P_k(x)\,dx\right)\tag{57}$$

По формуле Родрига

$$P_k(x)=\frac{1}{k!2^k}\frac{d^k}{dx^k}\left(x^2-1\right)^k.\tag{58}$$

При $k=0$ $P_0(x)=1$ и

$$\int\limits_{-1}^{1} f(x)\,P_0(x)\,dx = U_0\left(\int\limits_{0}^{1} P_0(x)\,dx - \int\limits_{-1}^{0} P_0(x)\,dx\right) = U_0\left(\int\limits_{0}^{1} dx - \int\limits_{-1}^{0} dx\right) = U_0\,(1-1) = 0. \quad (59)$$

При $k>0$

$$\int\limits_{-1}^{1} f(x)\,P_k(x)\,dx = \frac{U_0}{k!2^k}\left(\int\limits_{0}^{1} \frac{d^k}{dx^k}\left(x^2-1\right)^k dx - \int\limits_{-1}^{0} \frac{d^k}{dx^k}\left(x^2-1\right)^k dx\right) = \frac{U_0}{k!2^k}\left(\frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_0^1 - \frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_{-1}^0\right) =$$

$$= \frac{U_0}{k!2^k}\left(-\frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_{x=0} - \frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_{x=0}\right) = -\frac{U_0}{k!2^{k-1}}\,\frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_{x=0} \quad (60)$$

Вычислим отдельно

$$\left(x^2-1\right)^k = \sum_{m=0}^{k} C_k^m x^{2m}\,(-1)^{k-m}\,, \quad (61)$$

При $k=2n$

$$\frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k = \frac{d^{2n-1}}{dx^{2n-1}}\left(x^2-1\right)^{2n} = \frac{d^{2n-1}}{dx^{2n-1}}\sum_{m=0}^{2n} C_{2n}^m x^{2m}\,(-1)^{2n-m} =$$

$$= \frac{d^{2n-1}}{dx^{2n-1}}\left(\sum_{m=0}^{n-1} C_{2n}^m x^{2m}\,(-1)^{2n-m} + \sum_{m=n}^{2n} C_{2n}^m x^{2m}\,(-1)^{2n-m}\right) = \sum_{m=n}^{2n} C_{2n}^m \frac{(2m)!}{(2m-2n+1)!} x^{2m-2n+1}\,(-1)^{2n-m}\,, \quad (62)$$

$$\frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_{x=0} = 0, \quad (63)$$

$$\int\limits_{-1}^{1} f(x)\,P_{2n}(x)\,dx = -\frac{U_0}{k!2^{k-1}}\,\frac{d^{k-1}}{dx^{k-1}}\left(x^2-1\right)^k\Big|_{x=0} = 0. \quad (64)$$

При $k=2n-1$

$$\frac{d^{2n-2}}{dx^{2n-2}}\left(x^2-1\right)^{2n-1} = \frac{d^{2n-2}}{dx^{2n-2}}\sum_{m=0}^{2n-1} C_{2n-1}^m x^{2m}\,(-1)^{2n-1-m} =$$

$$= \frac{d^{2n-2}}{dx^{2n-2}}\left(\sum_{m=0}^{n-2} C_{2n-1}^m x^{2m}\,(-1)^{2n-1-m} + C_{2n-1}^{n-1} x^{2n-2}\,(-1)^n + \sum_{m=n}^{2n-1} C_{2n-1}^m x^{2m}\,(-1)^{2n-1-m}\right) = \quad (65)$$

$$= C_{2n-1}^{n-1}\,(2n-2)!\,(-1)^n + \sum_{m=n}^{2n-1} C_{2n-1}^m \frac{(2m)!}{(2m-2n+2)!} x^{2m-2n+2}\,(-1)^{2n-1-m}\,,$$

$$\frac{d^{2n-2}}{dx^{2n-2}}\left(x^2-1\right)^{2n-1}\Big|_{x=0} = C_{2n-1}^{n-1}\,(2n-2)!\,(-1)^n\,, \quad (66)$$

$$\int_{-1}^{1} f(x) P_{2n-1}(x)\,dx = -\frac{U_0}{(2n-1)!2^{2n-2}}\ \frac{d^{2n-2}}{dx^{2n-2}}\left(x^2-1\right)^{2n-1}\Big|_{x=0} = -\frac{U_0}{(2n-1)!2^{2n-2}} C_{2n-1}^{n-1}(2n-2)!\,(-1)^n =$$

$$= -\frac{(-1)^n U_0}{2^{2n-2}}\frac{(2n-2)!}{n!\,(n-1)!} \tag{67}$$

Итак,

$$C_{2n,0}^{6}\frac{2l^{2n}}{4n+1} = 0 \quad \Longrightarrow \quad C_{2n,0}^{6} = 0; \tag{68}$$

$$C_{2n-1,0}^{6}\frac{2l^{2n-1}}{4n-1} = -\frac{(-1)^n U_0}{2^{2n-2}}\frac{(2n-2)!}{n!\,(n-1)!}, \tag{69}$$

$$C_{2n-1,0}^{6} = -\frac{(-1)^n U_0}{(2l)^{2n-1}}\frac{(4n-1)(2n-2)!}{n!\,(n-1)!} \tag{70}$$

По формуле (51)

$$U = \sum_{m=0}^{\infty} C_{m0}^{6}\rho^m P_m(\cos\theta) = \sum_{n=0}^{\infty}\left(C_{2n,0}^{6}\rho^{2n}P_{2n}(\cos\theta) + C_{2n-1,0}^{6}\rho^{2n-1}P_{2n-1}(\cos\theta)\right) =$$

$$= -U_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2l)^{2n-1}}\frac{(4n-1)(2n-2)!}{n!\,(n-1)!}\rho^{2n-1}P_{2n-1}(\cos\theta) \tag{71}$$