Преобразование градиента, дивергенции и ротора в криволинейные координаты.

1 Определения

Замена $(x,y,z) \longrightarrow (p,q,s)$

$$\left\{ \begin{array}{l} x = x\,(p,q,s) \\ y = y\,(p,q,s) \\ z = z\,(p,q,s) \end{array} \right. \tag{1}$$

Радиус-вектор $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$

$$\vec{r}'_p = \vec{i}x'_p + \vec{j}y'_p + \vec{k}z'_p \tag{2}$$

Коэффициенты Ламэ:

$$H_p \equiv \left|\vec{r}'_p\right| = \left(x'_p\right)^2 + \left(y'_p\right)^2 + \left(z'_p\right)^2 \tag{3}$$

$$\vec{e}_p \equiv \frac{\vec{r}'_p}{H_p}, \qquad |\vec{e}_p| = 1 \tag{4}$$

Аналогично,

$$H_q \equiv \left|\vec{r}'_q\right|, \quad H_s \equiv \left|\vec{r}'_s\right| \tag{5}$$

$$\vec{e}_q \equiv \frac{\vec{r}'_q}{H_q}, \quad \vec{e}_s \equiv \frac{\vec{r}'_s}{H_s}$$

Векторы $\vec{e}_p, \vec{e}_q, \vec{e}_s$ – орты новой системы координат. У *ортогональных* систем они имеют такие свойства:

$$(\vec{e}_p \cdot \vec{e}_p) = 1, \quad (\vec{e}_q \cdot \vec{e}_q) = 1, \quad (\vec{e}_s \cdot \vec{e}_s) = 1, \tag{6}$$

$$(\vec{e}_p \cdot \vec{e}_q) = 0, \quad (\vec{e}_p \cdot \vec{e}_s) = 0, \quad (\vec{e}_q \cdot \vec{e}_s) = 0, \tag{7}$$

$$[\vec{e}_p \times \vec{e}_q] = \vec{e}_s, \quad [\vec{e}_q \times \vec{e}_s] = \vec{e}_p, \quad [\vec{e}_s \times \vec{e}_p] = \vec{e}_q. \tag{8}$$

2 Градиент

Продифференцируем (1) по x, y и z:

$$\left\{ \begin{array}{l} 1 = x'_p p'_x + x'_q q'_x + x'_s s'_x \\ 0 = y'_p p'_x + y'_q q'_x + y'_s s'_x \\ 0 = z'_p p'_x + z'_q q'_x + z'_s s'_x \end{array} \right. \qquad \left\{ \begin{array}{l} 0 = x'_p p'_y + x'_q q'_y + x'_s s'_y \\ 1 = y'_p p'_y + y'_q q'_y + y'_s s'_y \\ 0 = z\prime_p p'_y + z'_q q'_y + z'_s s'_y \end{array} \right. \qquad \left\{ \begin{array}{l} 0 = x'_p p'_z + x'_q q'_z + x'_s s'_z \\ 0 = y'_p p'_z + y'_q q'_z + y'_s s'_z \\ 1 = z'_p p'_z + z'_q q'_z + z'_s s'_z \end{array} \right. \tag{9}$$

По правилу Крамера, из первой системы

$$p'_x = \frac{\left| \begin{array}{ccc} 1 & x'_q & x'_s \\ 0 & y'_q & y'_s \\ 0 & z'_q & z'_s \end{array} \right|}{\left| \begin{array}{ccc} x'_p & x'_q & x'_s \\ y'_p & y'_q & y'_s \\ z'_p & z'_q & z'_s \end{array} \right|} = \frac{\left(\vec{i}, \vec{r}'_q, \vec{r}'_s\right)}{\left(\vec{r}'_p, \vec{r}'_q, \vec{r}'_s\right)} = \frac{H_q H_s \left(\vec{i}, \vec{e}_q, \vec{e}_s\right)}{H_p H_q H_s \left(\vec{e}_p, \vec{e}_q, \vec{e}_s\right)} = \frac{1}{H_p} \frac{\vec{i} \cdot [\vec{e}_q \times \vec{e}_s]}{\vec{e}_p \cdot [\vec{e}_q \times \vec{e}_s]} = \frac{1}{H_p} \frac{\vec{i} \cdot \vec{e}_p}{\vec{e}_p \cdot \vec{e}_p} = \frac{\vec{i} \cdot \vec{e}_p}{H_p}. \tag{10}$$

Аналогично, из второй системы в (9) получаем

$$p'_y = \frac{\begin{vmatrix} 0 & x'_q & x'_s \\ 1 & y'_q & y'_s \\ 0 & z'_q & z'_s \end{vmatrix}}{\begin{vmatrix} x'_p & x'_q & x'_s \\ y'_p & y'_q & y'_s \\ z'_p & z'_q & z'_s \end{vmatrix}} = \frac{\vec{j}\cdot\vec{e}_p}{H_p}, \qquad p'_z = \frac{\begin{vmatrix} 0 & x'_q & x'_s \\ 0 & y'_q & y'_s \\ 1 & z'_q & z'_s \end{vmatrix}}{\begin{vmatrix} x'_p & x'_q & x'_s \\ y'_p & y'_q & y'_s \\ z'_p & z'_q & z'_s \end{vmatrix}} = \frac{\vec{k}\cdot\vec{e}_p}{H_p}. \tag{11}$$

Так как для всякого вектора $\vec{a} = \vec{i}a_x + \vec{j}a_y + \vec{k}a_z = \vec{i}\left(\vec{i}\cdot\vec{a}\right) + \vec{j}\left(\vec{j}\cdot\vec{a}\right) + \vec{k}\left(\vec{k}\cdot\vec{a}\right)$,

$$\operatorname{grad} p = \vec{i}p'_x + \vec{j}p'_y + \vec{k}p'_z = \vec{i}\frac{\vec{i}\cdot\vec{e}_p}{H_p} + \vec{j}\frac{\vec{j}\cdot\vec{e}_p}{H_p} + \vec{k}\frac{\vec{k}\cdot\vec{e}_p}{H_p} = \frac{\vec{e}_p}{H_p}. \tag{12}$$

Решая системы (9) относительно производных других координат, тем же путём находим, что

$$\operatorname{grad} q = \frac{\vec{e}_q}{H_q}, \quad \operatorname{grad} s = \frac{\vec{e}_s}{H_s}. \tag{13}$$

Градиент произвольной функции $u\,(p, q, s)$ находится отсюда мгновенно:

$$\operatorname{grad} u = u'_p \operatorname{grad} p + u'_q \operatorname{grad} q + u'_s \operatorname{grad} s = \frac{u'_p}{H_p}\vec{e}_p + \frac{u'_q}{H_q}\vec{e}_q + \frac{u'_s}{H_s}\vec{e}_s \tag{14}$$

3 Вспомогательные формулы. Свойства производных $\vec{e}$

3.1

$$\frac{\partial}{\partial\gamma}\,|\vec{e}_\alpha\cdot\vec{e}_\beta = \delta_{\alpha\beta} \tag{15}$$

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha + \vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\beta = 0 \tag{16}$$

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\gamma}\vec{e}_\alpha = -\vec{e}_\alpha\cdot\frac{\partial}{\partial\gamma}\vec{e}_\beta, \tag{17}$$

где α, β и γ – произвольные координаты из новой системы координат

3.2

В частности, когда $\alpha = \beta$,

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\gamma}\vec{e}_\beta = -\vec{e}_\beta\cdot\frac{\partial}{\partial\gamma}\vec{e}_\beta, \tag{18}$$

$\gamma = \alpha$

$$\vec{e}_\beta\cdot\frac{\partial}{\partial\alpha}\vec{e}_\beta = 0, \tag{19}$$

3.3

$$\vec{r}''_{\alpha\beta} = \vec{r}''_{\beta\alpha} \tag{20}$$

$$\frac{\partial}{\partial\beta}\left(H_\alpha\vec{e}_\alpha\right) = \frac{\partial}{\partial\alpha}\left(H_\beta\vec{e}_\beta\right) \tag{21}$$

$$\frac{\partial}{\partial\beta}H_\alpha\vec{e}_\alpha + H_\alpha\frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{\partial}{\partial\alpha}H_\beta\vec{e}_\beta + H_\beta\frac{\partial}{\partial\alpha}\vec{e}_\beta\bigg| \cdot \vec{e}_\beta \tag{22}$$

В силу (19)

$$H_\alpha\vec{e}_\beta \cdot \frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{\partial}{\partial\alpha}H_\beta, \tag{23}$$

$$\vec{e}_\beta \cdot \frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{1}{H_\alpha}\frac{\partial}{\partial\alpha}H_\beta, \tag{24}$$

где $\alpha \neq \beta$.

3.4

Теперь умножим

$$\frac{\partial}{\partial\beta}H_\alpha\vec{e}_\alpha + H_\alpha\frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{\partial}{\partial\alpha}H_\beta\vec{e}_\beta + H_\beta\frac{\partial}{\partial\alpha}\vec{e}_\beta\bigg| \cdot \vec{e}_\gamma \tag{25}$$

$(\gamma \neq \alpha,\ \gamma \neq \beta)$

$$H_\alpha\vec{e}_\gamma \cdot \frac{\partial}{\partial\beta}\vec{e}_\alpha = H_\beta\vec{e}_\gamma \cdot \frac{\partial}{\partial\alpha}\vec{e}_\beta \tag{26}$$

$$\vec{e}_\gamma \cdot \frac{\partial}{\partial\beta}\vec{e}_\alpha = \frac{H_\beta}{H_\alpha}\vec{e}_\gamma \cdot \frac{\partial}{\partial\alpha}\vec{e}_\beta \tag{27}$$

4 Дивергенция

Для поля $\vec{F} = \vec{i}P + \vec{j}Q + \vec{k}R$ дивергенция примет вид

$$\begin{gathered}\operatorname{div}\vec{F} = P'_x + Q'_y + R'_z = \\ = P'_p p'_x + P'_q q'_x + P'_s s'_x + Q'_p p'_y + Q'_q q'_y + Q'_s s'_y + R'_p p'_z + R'_q q'_z + R'_s s'_z = \\ = \left(P'_p p'_x + Q'_p p'_y + R'_p p'_z\right) + \left(P'_q q'_x + Q'_q q'_y + R'_q q'_z\right) + \left(P'_s s'_x + Q'_s s'_y + R'_s s'_z\right) = \\ = \vec{F}'_p \cdot \operatorname{grad} p + \vec{F}'_q \cdot \operatorname{grad} q + \vec{F}'_s \cdot \operatorname{grad} s = \frac{\vec{F}'_p \cdot \vec{e}_p}{H_p} + \frac{\vec{F}'_q \cdot \vec{e}_q}{H_q} + \frac{\vec{F}'_s \cdot \vec{e}_s}{H_s}\end{gathered} \tag{28}$$

В локальном базисе$\vec{F} = F_p\vec{e}_p + F_q\vec{e}_q + F_s\vec{e}_s$

$$\begin{gathered}\vec{F}'_p \cdot \vec{e}_p = \frac{\partial}{\partial p}\left(F_p\vec{e}_p + F_q\vec{e}_q + F_s\vec{e}_s\right) \cdot \vec{e}_p = \\ = \left(\frac{\partial}{\partial p}F_p\vec{e}_p + \frac{\partial}{\partial p}F_q\vec{e}_q + \frac{\partial}{\partial p}F_s\vec{e}_s + F_p\frac{\partial}{\partial p}\vec{e}_p + F_q\frac{\partial}{\partial p}\vec{e}_q + F_s\frac{\partial}{\partial p}\vec{e}_s\right) \cdot \vec{e}_p =\end{gathered} \tag{29}$$

по (24)

$$= \frac{\partial}{\partial p} F_p + F_q \vec{e}_p \cdot \frac{\partial}{\partial p} \vec{e}_q + F_s \vec{e}_p \cdot \frac{\partial}{\partial p} \vec{e}_s = \frac{\partial}{\partial p} F_p + F_q \frac{1}{H_q} \frac{\partial}{\partial q} H_p + F_s \frac{1}{H_s} \frac{\partial}{\partial s} H_p =$$

$$= \frac{1}{H_q H_s} \left(H_q H_s \frac{\partial}{\partial p} F_p + F_q H_s \frac{\partial}{\partial q} H_p + F_s H_q \frac{\partial}{\partial s} H_p \right).$$

Так же получим, что

$$\vec{F}_q' \cdot \vec{e}_q = \frac{1}{H_p H_s} \left(H_p H_s \frac{\partial}{\partial q} F_q + F_p H_s \frac{\partial}{\partial p} H_q + F_s H_p \frac{\partial}{\partial s} H_q \right), \tag{30}$$

$$\vec{F}_s' \cdot \vec{e}_s = \frac{1}{H_p H_q} \left(H_p H_q \frac{\partial}{\partial s} F_s + F_p H_q \frac{\partial}{\partial p} H_s + F_q H_p \frac{\partial}{\partial q} H_s \right). \tag{31}$$

Тогда

$$\begin{gathered} \operatorname{div} \vec{F} = \frac{\vec{F}_p' \cdot \vec{e}_p}{H_p} + \frac{\vec{F}_q' \cdot \vec{e}_q}{H_q} + \frac{\vec{F}_s' \cdot \vec{e}_s}{H_s} = \\ = \frac{1}{H_q H_s H_p} \left(H_q H_s \frac{\partial}{\partial p} F_p + F_q H_s \frac{\partial}{\partial q} H_p + F_s H_q \frac{\partial}{\partial s} H_p \right) + \\ \frac{1}{H_p H_s H_q} \left(H_p H_s \frac{\partial}{\partial q} F_q + F_p H_s \frac{\partial}{\partial p} H_q + F_s H_p \frac{\partial}{\partial s} H_q \right) + \\ \frac{1}{H_p H_q H_s} \left(H_p H_q \frac{\partial}{\partial s} F_s + F_p H_q \frac{\partial}{\partial p} H_s + F_q H_p \frac{\partial}{\partial q} H_s \right) = \\ = \frac{1}{H_p H_q H_s} \left(H_q H_s \frac{\partial}{\partial p} F_p + F_p H_s \frac{\partial}{\partial p} H_q + F_p H_q \frac{\partial}{\partial p} H_s + \right. \\ + H_p H_s \frac{\partial}{\partial q} F_q + F_q H_s \frac{\partial}{\partial q} H_p + F_q H_p \frac{\partial}{\partial q} H_s + \\ \left. H_p H_q \frac{\partial}{\partial s} F_s + F_s H_q \frac{\partial}{\partial s} H_p + F_s H_p \frac{\partial}{\partial s} H_q \right) = \\ = \frac{1}{H_p H_q H_s} \left[\frac{\partial}{\partial p} \left(H_q H_s F_p \right) + \frac{\partial}{\partial q} \left(H_p H_s F_q \right) + \frac{\partial}{\partial s} \left(H_p H_q F_s \right) \right] \end{gathered} \tag{32}$$

5 Ротор (без доказательства)

$$\operatorname{rot} \vec{F} = \frac{1}{H_p H_q H_s} \begin{vmatrix} H_p \vec{e}_p & \partial/\partial p & H_p F_p \\ H_q \vec{e}_q & \partial/\partial q & H_q F_q \\ H_s \vec{e}_s & \partial/\partial s & H_s F_s \end{vmatrix} \tag{33}$$