$$\dot{X} = AX + F \tag{1}$$

$$\dot{X}_0 = AX_0 \tag{2}$$

$$\dot{X}_1 = AX_1 + F \tag{3}$$

$$X = X_0 + X_1 \tag{4}$$

1 Полиномы и экспоненты

$$F = \begin{pmatrix} P_m^1(t) \\ \vdots \\ P_m^n(t) \end{pmatrix} e^{\gamma t} \quad \Longrightarrow \quad X_1 = \begin{pmatrix} Q_{m+s}^1(t) \\ \vdots \\ Q_{m+s}^n(t) \end{pmatrix} e^{\gamma t} \tag{5}$$

828)

$$\begin{cases} \dot{x} = 3x + 2y + 4e^{5t} \\ \dot{y} = x + 2y \end{cases} \tag{6}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 4e^{5t} \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} e^{5t} \tag{7}$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-3)(\lambda-2) - 2 = \lambda^2 - 5\lambda + 4 = \lambda(\lambda-4) - (\lambda-4) = (\lambda-1)(\lambda-4) = 0 \tag{8}$$

$$\lambda_1 = 1 \qquad \lambda_2 = 4 \tag{9}$$

1)

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Longrightarrow \quad a + b = 0 \quad \Longrightarrow \quad b = -a \tag{10}$$

$$v_1 = \begin{pmatrix} a \\ -a \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} a \tag{11}$$

2)

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Longrightarrow \quad a = 2b \tag{12}$$

$$v_2 = \begin{pmatrix} 2b \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} b \tag{13}$$

$$X_0 = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} \tag{14}$$

$$\begin{cases} x_0 = C_1 e^t + 2C_2 e^{4t} \\ y_0 = -C_1 e^t + C_2 e^{4t} \end{cases} \tag{15}$$

X_1:

$$F = \begin{pmatrix} 4 \\ 0 \end{pmatrix} e^{5t} \tag{16}$$

$$m = 0, \quad \gamma = 5, \quad s = 0, \quad m + s = 0 \tag{17}$$

$$X_1 = \begin{pmatrix} a \\ b \end{pmatrix} e^{5t} \tag{18}$$

$$\begin{cases} x_1 = ae^{5t} \\ y_1 = be^{5t} \end{cases} \tag{19}$$

$$\begin{cases} 5ae^{5t} = 3ae^{5t} + 2be^{5t} + 4e^{5t} \\ 5be^{5t} = ae^{5t} + 2be^{5t} \end{cases} \implies \begin{cases} 5a = 3a + 2b + 4 \\ 5b = a + 2b \end{cases} \implies \begin{cases} a - b = 2 \\ -a + 3b = 0 \end{cases} \tag{20}$$

$$a = 3, \quad b = 1 \tag{21}$$

$$\begin{cases} x_1 = 3e^{5t} \\ y_1 = e^{5t} \end{cases} \tag{22}$$

Ответ:

$$X = X_0 + X_1 \tag{23}$$

$$\begin{cases} x = C_1e^t + 2C_2e^{4t} + 3e^{5t} \\ y = -C_1e^t + C_2e^{4t} + e^{5t} \end{cases} \tag{24}$$

(829, 831, 826)

2 Полиномы, экспоненты, синусы и косинусы

$$F = \left[\begin{pmatrix} P_m^1(t) \\ \vdots \\ P_m^n(t) \end{pmatrix} \cos(\beta t) + \begin{pmatrix} Q_m^1(t) \\ \vdots \\ Q_m^n(t) \end{pmatrix} \sin(\beta t) \right] e^{\alpha t} \implies X_1 = \left[\begin{pmatrix} S_{m+s}^1(t) \\ \vdots \\ S_{m+s}^n(t) \end{pmatrix} \cos(\beta t) + \begin{pmatrix} T_{m+s}^1(t) \\ \vdots \\ T_{m+s}^n(t) \end{pmatrix} \sin(\beta t) \right] e^{\alpha t} \tag{25}$$

827)

$$\begin{cases} \dot{x} = y - 5\cos t \\ \dot{y} = 2x + y \end{cases} \tag{26}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} -5\cos t \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \cos t \tag{27}$$

X_0:

$$\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda(\lambda - 1) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 1)(\lambda + 1) - (\lambda + 1) = (\lambda + 1)(\lambda - 2) = 0 \tag{28}$$

$$\lambda_1 = -1 \qquad \lambda_2 = 2 \tag{29}$$

1)

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \implies a + b = 0 \implies b = -a \tag{30}$$

$$v_1 = \begin{pmatrix} a \\ -a \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} a \tag{31}$$

2)

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \implies 2a - b = 0 \implies b = 2a \tag{32}$$

$$v_2 = \begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} a \tag{33}$$

$$X_0 = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} \tag{34}$$

$$\begin{cases} x_0 = C_1 e^{-t} + C_2 e^{2t} \\ y_0 = -C_1 e^{-t} + 2C_2 e^{2t} \end{cases} \tag{35}$$

X_1:

$$F = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \cos t \tag{36}$$

$$m = 0, \quad \beta = 1, \quad \alpha = 0 \tag{37}$$

$$\alpha + i\beta = i \quad \Longrightarrow \quad s = 0 \tag{38}$$

$$m + s = 0 \tag{39}$$

$$X_1 = \begin{pmatrix} a \\ b \end{pmatrix} \cos t + \begin{pmatrix} c \\ d \end{pmatrix} \sin t \tag{40}$$

$$\begin{cases} x_1 = a\cos t + c\sin t \\ y_1 = b\cos t + d\sin t \end{cases} \tag{41}$$

подставляем:

$$\begin{cases} -a\sin t + c\cos t = b\cos t + d\sin t - 5\cos t \\ -b\sin t + d\cos t = 2a\cos t + 2c\sin t + b\cos t + d\sin t \end{cases} \tag{42}$$

$t = 0$:

$$\begin{cases} c = b - 5 \\ d = 2a + b \end{cases} \tag{43}$$

$t = \frac{\pi}{2}$:

$$\begin{cases} -a = d \\ -b = 2c + d \end{cases} \tag{44}$$

$$\begin{cases} c = b - 5 \\ d = 2a + b \\ -a = d \\ -b = 2c + d \end{cases} \tag{45}$$

$$a = -1, \quad b = 3, \quad c = -2, \quad d = 1 \tag{46}$$

$$\begin{cases} x_1 = -\cos t - 2\sin t \\ y_1 = 3\cos t + \sin t \end{cases} \tag{47}$$

Ответ:

$$\begin{cases} x = C_1 e^{-t} + C_2 e^{2t} - \cos t - 2\sin t \\ y = -C_1 e^{-t} + 2C_2 e^{2t} + 3\cos t + \sin t \end{cases} \tag{48}$$

(834)

3 Неопределённые коэффициенты

$$\dot{X} - AX = F \tag{49}$$

$$\dot{X}_0 - AX_0 = 0 \tag{50}$$

$$X_0 = \sum_{k=1}^{n} C_k X_{0k}, \qquad \dot{X}_{0k} - AX_{0k} = 0 \tag{51}$$

ищем в виде

$$X = \sum_{k=1}^{n} \varphi_k(t) X_{0k} \tag{52}$$

$$\dot{X} = \sum_{k=1}^{n} \dot{\varphi}_k(t) X_{0k} + \sum_{k=1}^{n} \varphi_k(t) \dot{X}_{0k} \tag{53}$$

подставляем в систему

$$\sum_{k=1}^{n} \dot{\varphi}_k(t) X_{0k} + \sum_{k=1}^{n} \varphi_k(t) \dot{X}_{0k} - A\sum_{k=1}^{n} \varphi_k(t) X_{0k} = F \tag{54}$$

$$\sum_{k=1}^{n} \dot{\varphi}_k(t) X_{0k} + \sum_{k=1}^{n} \varphi_k(t) \left(\dot{X}_{0k} - AX_{0k}\right) = F \tag{55}$$

$$\sum_{k=1}^{n} \dot{\varphi}_k(t) X_{0k} = F \tag{56}$$

847)

$$\begin{cases} \dot{x} = 2y - x \\ \dot{y} = 4y - 3x + \frac{e^{3t}}{e^{2t}+1} \end{cases} \tag{57}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ \frac{e^{3t}}{e^{2t}+1} \end{pmatrix} \tag{58}$$

X_0

$$\begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = (\lambda-4)(\lambda+1)+6 = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0 \tag{59}$$

$$\lambda_1 = 1, \qquad \lambda_2 = 2 \tag{60}$$

1)

$$\begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Longrightarrow \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{61}$$

$(a = 1)$

$$X_{01} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \tag{62}$$

2)

$$\begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \Longrightarrow \quad -3a + 2b = 0 \quad \Longrightarrow \quad v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{63}$$

$(a=2)$

$$X_{02}=\left(\begin{array}{c}2\\3\end{array}\right)e^{2t} \tag{64}$$

X

$$X=\varphi_1 X_{01}+\varphi_2 X_{02} \tag{65}$$

$$\dot\varphi_1 X_{01}+\dot\varphi_2 X_{02}=F \tag{66}$$

$$\left\{\begin{array}{l}\dot\varphi_1 e^t+2\dot\varphi_2 e^{2t}=0\\ \dot\varphi_1 e^t+3\dot\varphi_2 e^{2t}=\frac{e^{3t}}{e^{2t}+1}\end{array}\right. \tag{67}$$

(2)-(1):

$$\dot\varphi_2 e^{2t}=\frac{e^{3t}}{e^{2t}+1}\quad\Longrightarrow\quad\dot\varphi_2=\frac{e^t}{e^{2t}+1} \tag{68}$$

$$\dot\varphi_1=-2\dot\varphi_2 e^t=-2\frac{e^{2t}}{e^{2t}+1} \tag{69}$$

$$\varphi_1=-2\int\frac{e^{2t}dt}{e^{2t}+1}=-\ln\left(e^{2t}+1\right)+C_1\qquad\varphi_2=\int\frac{e^t dt}{e^{2t}+1}=\operatorname{arctg}e^t+C_2 \tag{70}$$

$$X=\left(-\ln\left(e^{2t}+1\right)+C_1\right)\left(\begin{array}{c}1\\1\end{array}\right)e^t+\left(\operatorname{arctg}e^t+C_2\right)\left(\begin{array}{c}2\\3\end{array}\right)e^{2t} \tag{71}$$

$$\left\{\begin{array}{l}x=\left(-\ln\left(e^{2t}+1\right)+C_1\right)e^t+2\left(\operatorname{arctg}e^t+C_2\right)e^{2t}\\ y=\left(-\ln\left(e^{2t}+1\right)+C_1\right)e^t+3\left(\operatorname{arctg}e^t+C_2\right)e^{2t}\end{array}\right. \tag{72}$$

(849)