1 Однородные уравнения

$u\left(x,y,z\right)$

$$Pu'_x + Qu'_y + Ru'_z = 0 \tag{1}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{2}$$

$$\left\{ \begin{array}{l} \varphi\left(x,y,z\right) = C_1 \\ \psi\left(x,y,z\right) = C_2 \end{array} \right. \tag{3}$$

$$U = F\left(\varphi\left(x,y,z\right), \psi\left(x,y,z\right)\right), \quad \forall F\left(u,v\right) \tag{4}$$

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$$\left(x-z\right)\frac{\partial u}{\partial x} + \left(y-z\right)\frac{\partial u}{\partial y} + 2z\frac{\partial u}{\partial z} = 0 \tag{5}$$

$$\frac{dx}{x-z} = \frac{dy}{y-z} = \frac{dz}{2z} \tag{6}$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = \varkappa \tag{7}$$

$$\left\{ \begin{array}{c} a_1 = \varkappa b_1 \\ a_2 = \varkappa b_2 \\ \ldots\ldots\ldots\ldots \\ a_n = \varkappa b_n \end{array} \right. \tag{8}$$

$$+\left\{ \begin{array}{c} a_1 = \varkappa b_1 | \cdot k_1 \\ a_2 = \varkappa b_2 | \cdot k_2 \\ \ldots\ldots\ldots\ldots \\ a_n = \varkappa b_n | \cdot k_n \end{array} \right. \tag{9}$$

$$a_1 k_1 + \ldots + a_n k_n = \varkappa b_1 k_1 + \ldots + \varkappa b_n k_n \tag{10}$$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = \varkappa = \frac{a_1 k_1 + \ldots + a_n k_n}{b_1 k_1 + \ldots + b_n k_n} \tag{11}$$

Вернёмся к уравнению

$$\frac{dx}{x-z} = \frac{dy}{y-z} = \frac{dz}{2z} = \frac{dx-dy}{x-y} = \frac{dy+dz}{y+z} \tag{12}$$

1)

$$\frac{dz}{2z} = \frac{dx-dy}{x-y} \tag{13}$$

$$\frac{1}{2}d\ln|z| = d\ln|x-y| \tag{14}$$

$$\ln|z| = 2\ln|x-y| + \tilde{C}_1 \tag{15}$$

$$\ln\left|\frac{z}{\left(x-y\right)^2}\right| = \tilde{C}_1 \tag{16}$$

$$\frac{z}{\left(x-y\right)^2} = C_1 \tag{17}$$

2)

$$\frac{dx-dy}{x-y} = \frac{dy+dz}{y+z} \tag{18}$$

$$d\ln|x-y| = d\ln|y+z| \tag{19}$$

$$\frac{x-y}{y+z} = C_2 \tag{20}$$

Ответ: $u = F\left(\frac{z}{(x-y)^2}, \frac{x-y}{y+z}\right)$

Проверка:

$$u'_x = F'_1\left(\frac{z}{\left(x-y\right)^2}\right)'_x + F'_2\left(\frac{x-y}{y+z}\right)'_x = -2\frac{z}{\left(x-y\right)^3}F'_1 + \frac{1}{y+z}F'_2 \tag{21}$$

$$u'_y = F'_1\left(\frac{z}{\left(x-y\right)^2}\right)'_y + F'_2\left(\frac{x-y}{y+z}\right)'_y = F'_1\frac{-2z}{\left(x-y\right)^3}\left(-1\right) + F'_2\left(\frac{x-y-z+z}{y+z}\right)'_y =$$

$$= \frac{2z}{\left(x-y\right)^3}F'_1 + F'_2\left(\frac{x+z}{y+z} - 1\right)'_y = \frac{2z}{\left(x-y\right)^3}F'_1 - \frac{x+z}{\left(y+z\right)^2}F'_2 \tag{22}$$

$$u'_z = F'_1\left(\frac{z}{(x-y)^2}\right)'_z + F'_2\left(\frac{x-y}{y+z}\right)'_z = \frac{1}{(x-y)^2}F'_1 - \frac{x-y}{(y+z)^2}F'_2 \tag{23}$$

$$(x-z)\,u'_x + (y-z)\,u'_y + 2zu'_z = (x-z)\left[-2\frac{z}{(x-y)^3}F'_1 + \frac{1}{y+z}F'_2\right] +$$

$$+(y-z)\left[\frac{2z}{(x-y)^3}F'_1 - \frac{x+z}{(y+z)^2}F'_2\right] + 2z\left[\frac{1}{(x-y)^2}F'_1 - \frac{x-y}{(y+z)^2}F'_2\right] =$$

$$= \left[-2z\,(x-z) + (y-z)\,2z + 2z\,(x-y)\right]\frac{F'_1}{(x-y)^3} + \tag{24}$$

$$+\left[(x-z)\,(y+z) - (x+z)\,(y-z) - (x-y)\,2z\right]\frac{F'_2}{(y+z)^2} =$$

$$= 2z\left[-x+z+y-z+x-y\right]\frac{F'_1}{(x-y)^3} +$$

$$+\left[y\,(x-z) + z\,(x-z) + z\,(x+z) - y\,(x+z) - 2z\,(x-y)\right]\frac{F'_2}{(y+z)^2} =$$

$$= \left[xy - zy + xz - z^2 + zx + z^2 - yx - yz - 2zx + 2zy\right]\frac{F'_2}{(y+z)^2} =$$

$$= \left[\left(z^2 - z^2 + xy - yx\right) + (2zy - zy - yz) + (xz + zx - 2zx)\right]\frac{F'_2}{(y+z)^2} = 0$$

Верно.

Задание: №1169.

2 Неоднородные уравнения

$z\,(x,y)$

$$Pz'_x + Qz'_y = R \tag{25}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{26}$$

$$\begin{cases} \varphi\,(x,y,z) = C_1 \\ \psi\,(x,y,z) = C_2 \end{cases} \tag{27}$$

$$U = F\left\{\varphi\left[x,y,z\,(x,y)\right],\psi\left[x,y,z\,(x,y)\right]\right\} = C \tag{28}$$

$$\varphi\,(x,y,z) = f\,(\psi\,(x,y,z))\,, \quad \forall f\,(u) \tag{29}$$

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$$x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = x^2y + z \tag{30}$$

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{dz}{x^2y+z} \tag{31}$$

$$\frac{dx}{x} = \frac{dy}{2y}, \quad 2\ln|x| - \ln|y| = \tilde{C}_1, \quad \ln\frac{x^2}{|y|} = \tilde{C}_1, \quad \frac{x^2}{y} = C_1 \tag{32}$$

$$\frac{dy}{2y} = \frac{dz}{C_1y^2+z}, \qquad \frac{C_1y^2+z}{2y} = \frac{dz}{dy} \tag{33}$$

$$\frac{dz}{dy} - \frac{z}{2y} = \frac{C_1}{2}y \tag{34}$$

линейное

$$\frac{dz_0}{dy} - \frac{z_0}{2y} = 0, \quad \frac{dz_0}{z_0} = \frac{dy}{2y}, \quad z_0 = \sqrt{|y|} \tag{35}$$

$$z = z_0z_1 = \sqrt{|y|}z_1 \tag{36}$$

$$\operatorname{sgn} y\frac{z_1}{2\sqrt{|y|}} + \sqrt{|y|}z'_1 - \frac{\sqrt{|y|}z_1}{2y} = \frac{C_1}{2}y\,\Bigg|\,\frac{1}{\sqrt{|y|}} \tag{37}$$

$$\operatorname{sgn} y\frac{z_1}{2\sqrt{|y|}^2} + z'_1 - \frac{z_1}{2y} = \frac{C_1y}{2\sqrt{|y|}} = \operatorname{sgn} y\frac{C_1\,|y|}{2\sqrt{|y|}} = \operatorname{sgn} y\frac{C_1}{2}\sqrt{|y|} \tag{38}$$

$$z'_1 = \operatorname{sgn} y\frac{C_1}{2}\sqrt{|y|}, \quad z_1 = \frac{C_1}{3}\,|y|^{3/2} + C_2, \quad z = \sqrt{|y|}z_1 = \sqrt{|y|}\left(\frac{C_1}{3}\,|y|^{3/2} + C_2\right) =$$

$$= \frac{\frac{x^2}{y}}{3} y^2 + C_2\sqrt{|y|} = \frac{1}{3}x^2 y + C_2\sqrt{|y|} \tag{39}$$

$$\frac{3z - x^2 y}{3\sqrt{|y|}} = C_2 \tag{40}$$

$$\begin{cases} \varphi = \frac{x^2}{y} = C_1, \\ \psi = \frac{3z - x^2 y}{3\sqrt{|y|}} = C_2. \end{cases} \tag{41}$$

z связан с x и y уравнением

$$\psi = f(\varphi) \tag{42}$$

$$\frac{3z - x^2 y}{3\sqrt{|y|}} = f\left(\frac{x^2}{y}\right), \tag{43}$$

$\forall f$.

$$z = \sqrt{|y|} f\left(\frac{x^2}{y}\right) + \frac{1}{3}x^2 y \tag{44}$$

Для проверки продифференируем

$$z'_x = \sqrt{|y|} f'\left(\frac{x^2}{y}\right)\frac{2x}{y} + \frac{2}{3}xy \tag{45}$$

$$z'_y = \frac{\operatorname{sgn} y}{2\sqrt{|y|}} f\left(\frac{x^2}{y}\right) - \sqrt{|y|} f'\left(\frac{x^2}{y}\right)\frac{x^2}{y^2} + \frac{1}{3}x^2 \tag{46}$$

и подставим в исходное уравнение

$$x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = x^2 y + z \tag{47}$$

полученное:

$$\begin{gathered} x\left[\sqrt{|y|} f'\left(\frac{x^2}{y}\right)\frac{2x}{y} + \frac{2}{3}xy\right] + 2y\left[\frac{\operatorname{sgn} y}{2\sqrt{|y|}} f\left(\frac{x^2}{y}\right) - \sqrt{|y|} f'\left(\frac{x^2}{y}\right)\frac{x^2}{y^2} + \frac{1}{3}x^2\right] - \left[x^2 y + \sqrt{|y|} f\left(\frac{x^2}{y}\right) + \frac{1}{3}x^2 y\right] = \\ = \frac{2x^2}{y}\sqrt{|y|} f'\left(\frac{x^2}{y}\right) - \frac{2x^2}{y}\sqrt{|y|} f'\left(\frac{x^2}{y}\right) + 2y\frac{\operatorname{sgn} y}{2\sqrt{|y|}} f\left(\frac{x^2}{y}\right) + \frac{4}{3}x^2 y - \frac{4}{3}x^2 y - \sqrt{|y|} f\left(\frac{x^2}{y}\right) = \\ = \frac{|y|}{\sqrt{|y|}} f\left(\frac{x^2}{y}\right) - \sqrt{|y|} f\left(\frac{x^2}{y}\right) = 0, \end{gathered} \tag{48}$$

что и т.д.

Задание: №1174.

3 Доп. условия

Пример: №1196 Найти поверхность, удовлетворяющую уравнению

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - xy, \tag{49}$$

и проходящую через линию

$$x = 2, \quad z = y^2 + 1. \tag{50}$$

Составим систему

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - xy} \tag{51}$$

и начнём её решать.

$$\frac{dx}{x} = \frac{dy}{y}, \quad \frac{dx}{x} - \frac{dy}{y} = 0, \quad \ln|x| - \ln|y| = \tilde{C}_1, \quad \frac{x}{y} = C_1 \tag{52}$$

$$\frac{dy}{y} = \frac{dz}{z - xy} = \frac{dz}{z - C_1 y^2} \tag{53}$$

$$\left(z - C_1 y^2\right) dy = ydz, \quad -C_1 y^2 dy = ydz - zdy \tag{54}$$

$$-C_1 dy = \frac{ydz - zdy}{y^2} = d\frac{z}{y}, \quad d\frac{z}{y} + C_1 dy = 0 \tag{55}$$

$$\frac{z}{y} + \frac{x}{y}y = C_2 \tag{56}$$

$$\begin{cases} \varphi = \frac{x}{y} = C_1 \\ \psi = \frac{z}{y} + x = C_2 \end{cases} \tag{57}$$

Если поверхность $\psi = f(\varphi)$ проходит через линию $x = 2$, $z = y^2 + 1$, то на этой линии

$$\begin{cases} \varphi = \frac{2}{y} \\ \psi = \frac{y^2+1}{y} + 2 \end{cases} \tag{58}$$

$$y = \frac{2}{\varphi}, \quad \frac{\left(\frac{2}{\varphi}\right)^2 + 1}{\frac{2}{\varphi}} + 2 = \psi, \tag{59}$$

$$\frac{\frac{4}{\varphi} + \varphi}{2} + 2 = \psi \tag{60}$$

$$\frac{4}{\varphi} + \varphi - 2\psi + 4 = 0 \tag{61}$$

$$\frac{4}{\frac{x}{y}} + \frac{x}{y} - 2\left(\frac{z}{y} + x\right) + 4 = 0 \tag{62}$$

$$\frac{4y}{x} + \frac{x}{y} - 2\frac{z}{y} - 2x + 4 = 0. \tag{63}$$

Проверяем. На линии

$$x = 2, \quad z = y^2 + 1 \tag{64}$$

получается

$$2y + \frac{2}{y} - 2\frac{y^2+1}{y} = 0. \tag{65}$$

$$2y - 2y = 0. \tag{66}$$

Производные:

$$-\frac{4y}{x^2} + \frac{1}{y} - 2\frac{z'_x}{y} - 2 = 0, \qquad z'_x = -\frac{2y^2}{x^2} + \frac{1}{2} - y \tag{67}$$

$$\frac{4}{x} - \frac{x}{y^2} - 2\frac{z'_y}{y} + 2\frac{z}{y^2} = 0, \qquad z'_y = \frac{2y}{x} - \frac{x}{2y} + \frac{z}{y} \tag{68}$$

Подставляем в диф. уравнение:

$$x\left(-\frac{2y^2}{x^2} + \frac{1}{2} - y\right) + y\left(\frac{2y}{x} - \frac{x}{2y} + \frac{z}{y}\right) = z - xy, \tag{69}$$

$$-\frac{2y^2}{x} + \frac{x}{2} - xy + \frac{2y^2}{x} - \frac{x}{2} + z = z - xy, \tag{70}$$

$$-xy + z = z - xy, \tag{71}$$

и всё сходится.

Задание: №1194 и 1195