

1)

$$\sqrt{ax^2 + bx + c} = \pm\sqrt{ax} + z \quad (a > 0) \quad (1)$$

$$ax^2 + bx + c = ax^2 \pm 2\sqrt{ax}z + z^2 \quad (2)$$

$$bx \mp 2\sqrt{ax}z = z^2 - c \quad (3)$$

$$x = \frac{z^2 - c}{b \mp 2\sqrt{az}} \quad (4)$$

$$\begin{aligned} dx &= \frac{(2z)(b \mp 2\sqrt{az}) - (z^2 - c)(\mp 2\sqrt{a})}{(b \mp 2\sqrt{az})^2} dz = \frac{(2bz \mp 4\sqrt{a}z^2) \pm 2\sqrt{a}(z^2 - c)}{(b \mp 2\sqrt{az})^2} dz = \\ &= \mp 2 \frac{\sqrt{a}z^2 \pm bz + \sqrt{ac}}{(b \mp 2\sqrt{az})^2} dz \end{aligned} \quad (5)$$

2)

$$\sqrt{ax^2 + bx + c} = xz \pm \sqrt{c} \quad (c > 0) \quad (6)$$

$$ax^2 + bx + c = x^2z^2 \pm 2\sqrt{c}xz + c \quad (7)$$

$$ax + b = xz^2 \pm 2\sqrt{c}z \quad (8)$$

$$ax - xz^2 = \pm 2\sqrt{c}z - b \quad (9)$$

$$x = \frac{\pm 2\sqrt{c}z - b}{a - z^2} \quad (10)$$

$$\begin{aligned} dx &= \frac{(\pm 2\sqrt{c})(a - z^2) - (\pm 2\sqrt{c}z - b)(-2z)}{(a - z^2)^2} dz = \frac{\pm 2\sqrt{c}(a - z^2) + 2z(\pm 2\sqrt{c}z - b)}{(a - z^2)^2} dz = \\ &= \frac{\pm 2\sqrt{c}a \pm 2\sqrt{c}z^2 - 2zb}{(a - z^2)^2} dz = \pm 2 \frac{\sqrt{c}a + \sqrt{c}z^2 \mp zb}{(a - z^2)^2} dz \end{aligned} \quad (11)$$

3)

$$\sqrt{a(x - x_1)(x - x_2)} = z(x - x_1) \quad (b^2 - 4ac > 0) \quad (12)$$

$$a(x - x_1)(x - x_2) = z^2(x - x_1)^2 \quad (13)$$

$$ax - z^2x = ax_2 - x_1z^2 \quad (14)$$

$$x = \frac{ax_2 - x_1z^2}{a - z^2} \quad (15)$$

$$\begin{aligned} dx &= \frac{(-x_1z)(a - z^2) - (ax_2 - x_1z^2)(-2z)}{(a - z^2)^2} dz = 2z \frac{(ax_2 - x_1z^2) - x_1(a - z^2)}{(a - z^2)^2} dz = \\ &= 2z \frac{ax_2 - x_1z^2 - ax_1 + x_1z^2}{(a - z^2)^2} dz = 2az \frac{x_2 - x_1}{(a - z^2)^2} dz \end{aligned} \quad (16)$$

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$$\int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx \quad (17)$$

Замена:

$$\begin{aligned} \sqrt{x^2 + 3x + 2} &= x + z \\ z &= \sqrt{x^2 + 3x + 2} - x \end{aligned} \quad (18)$$

$$x^2 + 3x + 2 = x^2 + 2xz + z^2 \quad (19)$$

$$3x - 2xz = z^2 - 2 \quad (20)$$

$$x = \frac{z^2 - 2}{3 - 2z} \quad (21)$$

$$dx = \frac{2z(3 - 2z) + 2(z^2 - 2)}{(3 - 2z)^2} dz = \frac{6z - 4z^2 + 2z^2 - 4}{(3 - 2z)^2} dz = -2 \frac{z^2 - 3z + 2}{(3 - 2z)^2} dz = -2 \frac{(z - 2)(z - 1)}{(3 - 2z)^2} dz \quad (22)$$

Подставим:

$$\begin{aligned} \int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx &= -2 \int \frac{x - (x + z)(z - 2)(z - 1)}{x + x + z} \frac{(z - 2)(z - 1)}{(3 - 2z)^2} dz = 2 \int \frac{z}{2x + z} \frac{(z - 2)(z - 1)}{(3 - 2z)^2} dz = \\ &= 2 \int \frac{z}{2 \frac{z^2 - 2}{3 - 2z} + z} \frac{(z - 2)(z - 1)}{(3 - 2z)^2} dz = 2 \int \frac{z}{2 \frac{z^2 - 2}{3 - 2z} + z} \frac{(z - 2)(z - 1)}{(3 - 2z)} dz = \end{aligned} \quad (23)$$

$$= 2 \int \frac{z}{3z-4} \frac{(z-2)(z-1)}{(3-2z)} dz = -2 \int \frac{z(z-2)(z-1)}{(3z-4)(2z-3)} dz.$$

Рац. дробь:

$$\frac{z(z-2)(z-1)}{(3z-4)(2z-3)} = \frac{z^3 - 3z^2 + 2z}{6z^2 - 17z + 12} = \frac{1}{6} \frac{6z^3 - 18z^2 + 12z}{6z^2 - 17z + 12} = \frac{1}{6} \left( z - \frac{1}{6} - \frac{1}{6} \frac{17z - 12}{(3z-4)(2z-3)} \right) \quad (24)$$

$$\frac{17z - 12}{(3z-4)(2z-3)} = \frac{A}{3z-4} + \frac{B}{2z-3} = \frac{A(2z-3) + B(3z-4)}{(3z-4)(2z-3)} \quad (25)$$

$$17z - 12 = A(2z-3) + B(3z-4) = 2Az - 3A + 3Bz - 4B = (2A+3B)z - 3A - 4B \quad (26)$$

$$\begin{cases} 2A + 3B = 17 \\ -3A - 4B = -12 \end{cases} \quad \begin{cases} 2A + 3B = 17 \\ 3A + 4B = 12 \end{cases} \quad (27)$$

$$A + B = -5 \quad (28)$$

$$B = 27, \quad A = -5 - B = -32 \quad (29)$$

$$\frac{17z - 12}{(3z-4)(2z-3)} = \frac{-32}{3z-4} + \frac{27}{2z-3} \quad (30)$$

$$\frac{z(z-2)(z-1)}{(3z-4)(2z-3)} = \frac{1}{6} \left[ z - \frac{1}{6} - \frac{1}{6} \left( \frac{-32}{3z-4} + \frac{27}{2z-3} \right) \right] = \frac{1}{6} z - \frac{1}{36} + \frac{1}{9} \frac{8}{3z-4} - \frac{1}{4} \frac{3}{2z-3} \quad (31)$$

$$\begin{aligned} \int \frac{x - \sqrt{x^2 + 3x + 2}}{x + \sqrt{x^2 + 3x + 2}} dx &= -2 \int \frac{z(z-2)(z-1)}{(3z-4)(2z-3)} dz = -2 \int \left( \frac{1}{6} z - \frac{1}{36} + \frac{8}{9} \frac{1}{3z-4} - \frac{3}{4} \frac{1}{2z-3} \right) dz = \\ &= -\frac{1}{6} z^2 + \frac{1}{18} z - \frac{16}{27} \ln |3z-4| + \frac{3}{4} \ln |2z-3| + C = \end{aligned} \quad (32)$$

$$= -\frac{1}{6} \left( \sqrt{x^2 + 3x + 2} - x \right)^2 + \frac{1}{18} \left( \sqrt{x^2 + 3x + 2} - x \right) - \frac{16}{27} \ln \left| 3 \left( \sqrt{x^2 + 3x + 2} - x \right) - 4 \right| + \frac{3}{4} \ln \left| 2 \left( \sqrt{x^2 + 3x + 2} - x \right) - 3 \right| + C.$$