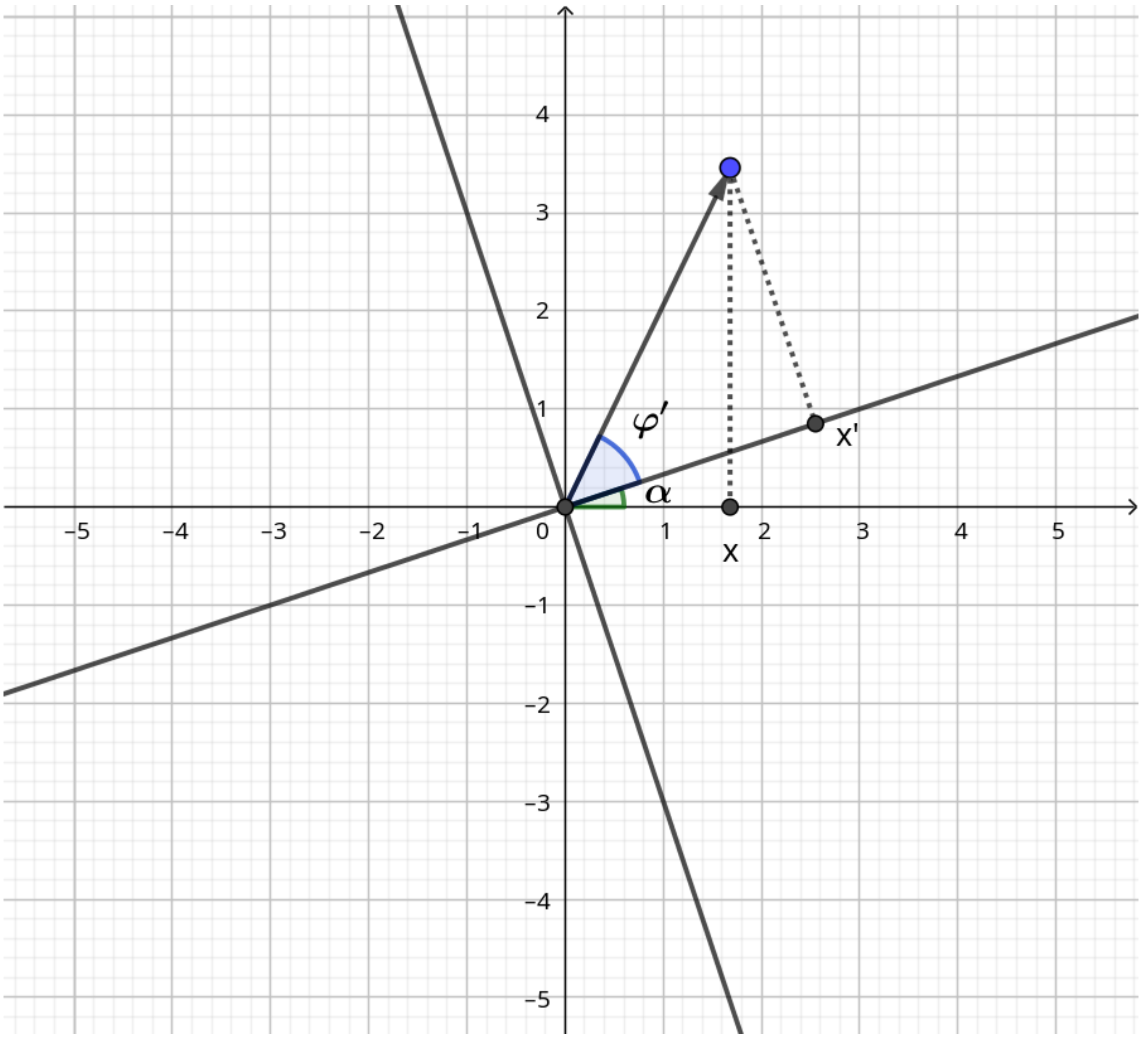
1 Теория

$$Ax^2 + 2Bxy + Cy^2 + ax + by + c = 0 \tag{1}$$



$$\begin{cases} x' = r' \cos \varphi' \\ y' = r' \sin \varphi' \end{cases} \tag{2}$$

$r = r'$, $\varphi = \varphi' + \alpha$

$$\begin{cases} x = r \cos \varphi = r' \cos(\varphi' + \alpha) = r' \cos \varphi' \cos \alpha - r' \sin \varphi' \sin \alpha = x' \cos \alpha - y' \sin \alpha \\ y = r \sin \varphi = r' \sin(\varphi' + \alpha) = r' \sin \varphi' \cos \alpha + r' \cos \varphi' \sin \alpha = y' \cos \alpha + x' \sin \alpha \end{cases} \tag{3}$$

Новая квадратичная часть:

$$\begin{gathered} Ax^2 + 2Bxy + Cy^2 = A(x' \cos \alpha - y' \sin \alpha)^2 + 2B(x' \cos \alpha - y' \sin \alpha)(y' \cos \alpha + x' \sin \alpha) + C(y' \cos \alpha + x' \sin \alpha)^2 = \\ = A\left(\underline{x'^2 \cos^2 \alpha} - 2x'y' \sin \alpha \cos \alpha + \underset{\sim}{y'^2 \sin^2 \alpha}\right) + \\ +2B\left(\underline{x'^2 \sin \alpha \cos \alpha} - \underset{\sim}{y'^2 \cos \alpha \sin \alpha} + x'y' \cos^2 \alpha - y'x' \sin^2 \alpha\right) + \\ +C\left(\underset{\sim}{y'^2 \cos^2 \alpha} + 2x'y' \cos \alpha \sin \alpha + \underline{x'^2 \sin^2 \alpha}\right) = \\ = \left(A \cos^2 \alpha + 2B \sin \alpha \cos \alpha + C \sin^2 \alpha\right) x'^2 + \left(A \sin^2 \alpha - 2B \cos \alpha \sin \alpha + C \cos^2 \alpha\right) y^2 + \\ + \left[-2A \sin \alpha \cos \alpha + 2B\left(\cos^2 \alpha - \sin^2 \alpha\right) + 2C \cos \alpha \sin \alpha\right] x'y' = \\ = \left(A \cos^2 \alpha + B \sin 2\alpha + C \sin^2 \alpha\right) x'^2 + \left(A \sin^2 \alpha - B \sin 2\alpha + C \cos^2 \alpha\right) y'^2 + \\ + (-A \sin 2\alpha + 2B \cos 2\alpha + C \sin 2\alpha) x'y' \end{gathered} \tag{4}$$

Потребуем:

$$-A\sin 2\alpha + 2B\cos 2\alpha + C\sin 2\alpha = 0 \left| \cdot \frac{1}{\sin 2\alpha} \right. \tag{5}$$

$\sin 2\alpha \neq 0,\ 2\alpha \neq \pi n,\ \alpha \neq \frac{\pi}{2}n$

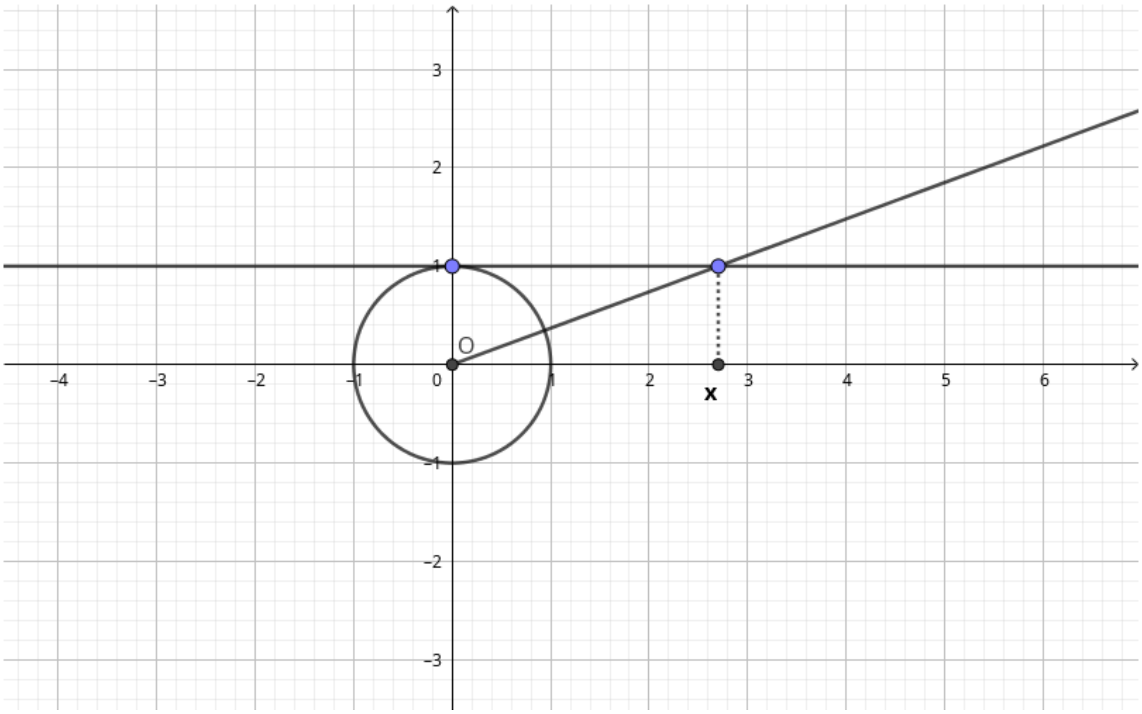
$$-A + 2B\operatorname{ctg} 2\alpha + C = 0 \tag{6}$$

$B \neq 0$

$$\operatorname{ctg}\ 2\alpha = \frac{A-C}{2B} \tag{7}$$

$A \neq C$

$$\operatorname{tg}\ 2\alpha = \frac{2B}{A-C} \tag{8}$$



$$2\alpha = \operatorname{arcctg}\frac{A-C}{2B} \in (0;\pi)\,, \qquad \alpha = \frac{1}{2}\operatorname{arcctg}\frac{A-C}{2B} \in \left(0;\frac{\pi}{2}\right) \tag{9}$$

2 Пример 1: школьная гипербола

$$y = \frac{1}{x}, \qquad xy - 1 = 0 \tag{10}$$

$$Ax^2 + 2Bxy + Cy^2 = xy \tag{11}$$

$$A = C = 0, \quad 2B = 1 \tag{12}$$

$$\alpha = \frac{1}{2}\operatorname{arcctg}\frac{A-C}{2B} = \frac{1}{2}\operatorname{arcctg}\frac{0}{1} = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4} \tag{13}$$

$$Ax^2 + 2Bxy + Cy^2 = xy = \tag{14}$$

$$= B\sin 2\alpha x'^2 - B\sin 2\alpha y^2 + 2B\cos 2\alpha x'y' = \frac{1}{2}\sin\frac{\pi}{2}x'^2 - \frac{1}{2}\sin\frac{\pi}{2}y'^2 + \cos\frac{\pi}{2}x'y' = \frac{1}{2}x'^2 - \frac{1}{2}y'^2$$

$$\frac{1}{2}x'^2 - \frac{1}{2}y'^2 - 1 = 0 \tag{15}$$

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1 \tag{16}$$

3 Пример 2:

$$2x^2 - 4xy + 5y^2 + 8x - 2y + 9 = 0 \tag{17}$$

$$A = 2, \quad B = -2, \quad C = 5, \quad \frac{A - C}{2B} = \frac{2 - 5}{-4} = \frac{3}{4} \tag{18}$$

$$\operatorname{ctg} 2\alpha = \frac{3}{4}, \quad 2\alpha \in \left(0; \frac{\pi}{2}\right) \tag{19}$$

$$\operatorname{tg} 2\alpha = \frac{4}{3} \tag{20}$$

триг. выражения

$$\operatorname{tg}^2 2\alpha + 1 = \frac{1}{\cos^2 2\alpha} \tag{21}$$

$$\cos^2 2\alpha = \frac{1}{\operatorname{tg}^2 2\alpha + 1} = \frac{1}{\left(\frac{4}{3}\right)^2 + 1} = \frac{9}{16 + 9} = \frac{9}{25}, \quad \cos 2\alpha = \frac{3}{5} \tag{22}$$

$$\sin^2 2\alpha = 1 - \cos^2 2\alpha = 1 - \frac{9}{25} = \frac{16}{25}, \quad \sin 2\alpha = \frac{4}{5} \tag{23}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} = \frac{1 + \frac{3}{5}}{2} = \frac{8}{10} = \frac{4}{5} \tag{24}$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} = \frac{1 - \frac{3}{5}}{2} = \frac{2}{10} = \frac{1}{5} \tag{25}$$

Квадратичная часть

$$\begin{gathered} Ax^2 + 2Bxy + Cy^2 = \\ = \left(A\cos^2\alpha + B\sin 2\alpha + C\sin^2\alpha\right)x'^2 + \left(A\sin^2\alpha - B\sin 2\alpha + C\cos^2\alpha\right)y'^2 + \\ + \left[(C - A)\sin 2\alpha + 2B\cos 2\alpha\right]x'y'; \end{gathered} \tag{26}$$

отсюда

$$\begin{gathered} 2x^2 - 4xy + 5y^2 = \\ = \left(2\cos^2\alpha - 2\sin 2\alpha + 5\sin^2\alpha\right)x'^2 + \left(2\sin^2\alpha + 2\sin 2\alpha + 5\cos^2\alpha\right)y'^2 + \\ + \left[3\sin 2\alpha - 4\cos 2\alpha\right]x'y' = \\ = \left(2\frac{4}{5} - 2\frac{4}{5} + 5\frac{1}{5}\right)x'^2 + \left(2\frac{1}{5} + 2\frac{4}{5} + 5\frac{4}{5}\right)y'^2 + \\ + \left[3\frac{4}{5} - 4\frac{3}{5}\right]x'y' = x'^2 + 6y'^2. \end{gathered} \tag{27}$$

Линейная

$$\cos\alpha = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}; \qquad \sin\alpha = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} \tag{28}$$

$$\begin{cases} x = x'\cos\alpha - y'\sin\alpha = x'\frac{2}{\sqrt{5}} - y'\frac{1}{\sqrt{5}} \\ y = y'\cos\alpha + x'\sin\alpha = y'\frac{2}{\sqrt{5}} + x'\frac{1}{\sqrt{5}} \end{cases} \tag{29}$$

$$\begin{gathered} 8x - 2y = 8\left(x'\frac{2}{\sqrt{5}} - y'\frac{1}{\sqrt{5}}\right) - 2\left(y'\frac{2}{\sqrt{5}} + x'\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}\left[8\left(2x' - y'\right) - 2\left(2y' + x'\right)\right] = \\ = \frac{1}{\sqrt{5}}\left(14x' - 12y'\right) = \frac{14}{\sqrt{5}}x' - \frac{12}{\sqrt{5}}y'. \end{gathered} \tag{30}$$

Всё вместе

$$\begin{gathered} 2x^2 - 4xy + 5y^2 + 8x - 2y + 9 = x'^2 + 6y'^2 + \frac{14}{\sqrt{5}}x' - \frac{12}{\sqrt{5}}y' + 9 = \\ = \left(x'^2 + 2\frac{7}{\sqrt{5}}x' + \frac{49}{5} - \frac{49}{5}\right) + 6\left(y'^2 - 2\frac{1}{\sqrt{5}}y' + \frac{1}{5} - \frac{1}{5}\right) + 9 = \\ = \left(x' + \frac{7}{\sqrt{5}}\right)^2 + 6\left(y' - \frac{1}{\sqrt{5}}\right)^2 + 9 - \frac{49}{5} - \frac{6}{5} = 0 \end{gathered} \tag{31}$$

$$\left(x' + \frac{7}{\sqrt{5}}\right)^2 + 6\left(y' - \frac{1}{\sqrt{5}}\right)^2 = 2 \tag{32}$$

$$\frac{\left(x' + \frac{7}{\sqrt{5}}\right)^2}{2} + \frac{\left(y' - \frac{1}{\sqrt{5}}\right)^2}{\frac{2}{6}} = 1 \tag{33}$$

$$\left(\frac{x' + \frac{7}{\sqrt{5}}}{\sqrt{2}}\right)^2 + \left(\frac{y' - \frac{1}{\sqrt{5}}}{\sqrt{\frac{1}{3}}}\right)^2 = 1 \tag{34}$$

Центр: $\left(-\frac{7}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$$\begin{cases} x = x'\frac{2}{\sqrt{5}} - y'\frac{1}{\sqrt{5}} = -\frac{7}{\sqrt{5}}\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}} = -\frac{14}{5} - \frac{1}{5} = -3 \\ y = y'\frac{2}{\sqrt{5}} + x'\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{2}{\sqrt{5}} - \frac{7}{\sqrt{5}}\frac{1}{\sqrt{5}} = \frac{2}{5} - \frac{7}{5} = -1 \end{cases} \tag{35}$$